Theoretical and Applied Economics
Volume XXII (2015), No. 4(605), Winter, pp. 99-104

Modelling optimal execution strategies for Algorithmic trading

Virgil DAMIAN
Bucharest University of Economic Studies, Romania
virgil.damian@fin.ase.ro

Abstract. *This paper is focused on the optimal execution of portfolio transactions considered as a stochastic optimal control problem. The main novelty of this work consists in a new methodology, introduced in Udrişte and Damian in 2011, for the stochastic optimal control problems, applied to Almgren and Chriss execution model. In addition to Udrişte (2015), this paper highlights our original ideas and certifies that the new above mentioned method is viable in this framework.*

Keywords: optimal execution strategies, algorithmic trading, cost of trading, order book, stochastic optimal control, stochastic differential systems.

JEL Classification: C02, C61, D53, G11, G12.

1. Introduction

The basic question in optimal liquidation problem is: *What is the schedule of executing a portfolio with x_0 shares?* The classical trade-off in this subject refers to that liquidating fast may be too expensive *versus* liquidating too slowly, the price may go down and we would have been better executing faster. With this respect, we need to find an *optimal* trading schedule. The recognized seminal work in this sense is the article Almgren and Chriss (2000), which defined the objective function as an execution cost that prompts agents to spread their executions while their risk aversion spurs them to trade fast. Another generation of models appeared along with the paper of Obizhaeva and Wang (2013)[(1)]. These models consider the execution cost in Almgren and Chriss framework linked to the order book dynamic and its resilience. The third generation of models applied to optimal liquidation was developed by Guéant, Lehalle and Fernandez-Tapia (2012) and uses the Avellaneda-Stoikov framework introduced in 2008.

In this paper, we resolve the Almgren-Chriss model using our new methodology in solving stochastic optimal control problems, introduced in Udrişte and Damian in 2011.

2. Stochastic optimal control theory

In Udrişte and Damian (2011), we proposed a new paradigm in approaching stochastic optimal control problems, in the sense of Pontryagin, which represents a relevant contribution in the existing (and various) related literature, the main idea being based on a geometric interpretation for the classical tools in stochastic calculus. This new methodology assumes the hypotheses that the optimal control is in the *interior* of the set of admissible controls and that the variational calculus techniques can be applied.

Stochastic optimal control problems have some common features: *(i)* there is a constraint diffusion system, which is described by an Itô stochastic differential system; *(ii)* there are some other constraints that the decisions and/or the state are subject to; *(iii)* there is a criterion that measures the performance of the decisions. The goal is to optimize the criterion (cost functional) by selecting a non-anticipative decision among the ones satisfying all the constraints.

Given a filtered probability space, satisfying the usual conditions, on which a $\mathbb{R}^d$ vector-valued Brownian motion process $(\boldsymbol{B}_t)_{t\in[0,T]}$ is defined, we consider a constraint as a controlled stochastic system:

$$dx_{i,t} = \mu_i(t, x_t, u_t)dt + \sum_{a=1}^{d} \sigma_{i,a}(t, x_t, u_t)dB_{a,t}, \tag{2.1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$, $\boldsymbol{\sigma} \in \mathbb{R}^{n\times d}$ are, respectively differentiable functions, and $\boldsymbol{x}_t = (x_{1,t}, \dots, x_{n,t}) \in \mathbb{R}^n$ is the solution of SDE (1). The stochastic process $(\boldsymbol{u}_t)_{t\in[0,T]} \in \mathbb{R}^k$ is called *control* and we require that it gives rise to a unique solution $\boldsymbol{x}_t = \boldsymbol{x}_t^{(u)}$ of SDE (1). Next, we introduce the *cost functional* as follows:

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, x_t, u_t)dt + \Psi(x_T)\right], \tag{2.2}$$

where f and Ψ are real-valued functions. The simplest stochastic optimal control problem can be stated as follows:

$$\text{Find } \max_{(\boldsymbol{u}_t)_{t\in[0,T]}} J(u(\cdot)) \quad \text{constrained by (1).} \tag{2.3}$$

In order to solve the problem (3), we introduce the *control Hamiltonian stochastic* functional, defined by:

$$\mathcal{H}(t, \boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{p}_t) = f(t, x_t, u_t)dt +$$
$$+\left[\sum_{i=1}^{n} p_{i,t}\cdot\mu_{i,t} - \sum_{i,j=1}^{n}\sum_{a,b=1}^{d} p_{i,t}\cdot\frac{\partial\sigma_{i,a}}{\partial x_j}\cdot\sigma_{j,b}\cdot\delta_{a,b}\right]dt, \tag{2.4}$$

where $\delta_{a,b}$ denotes the *Kronecker symbol*, defined by $\delta_{a,b} = 1$, if $a = b$ and $\delta_{a,b} = 0$ for $a \neq b$.

Theorem (Udrişte and Damian, 2011). *Suppose that the control problem (2.3), constrained by (2.1), has an interior optimal solution* $(\boldsymbol{u}_t^*)_{t\in[0,T]}$, *which determines the stochastic optimal evolution* $(\boldsymbol{x}_t)_{t\in[0,T]}$. *Let* $\mathcal{H}$ *be the Hamiltonian stochastic functional (2.4). Then there exists an adapted process* $(\boldsymbol{p}_t)_{t\in[0,T]} \in \mathbb{R}^n$ *(namely adjoint process) satisfying:*
(i) the initial stochastic differential system:

$$dx_{i,t} = \frac{\partial\mathcal{H}}{\partial p_i} + \sum_{i,j=1}^{n}\sum_{a,b=1}^{d}\frac{\partial\sigma_{i,a}}{\partial x_j}\cdot\sigma_{j,b}\cdot\delta_{a,b}\,dt + \sum_{a=1}^{d}\sigma_{i,a}\,dB_{a,t}; \tag{2.5}$$

(ii) the adjoint linear stochastic differential system:

$$dp_{i,t} = -\frac{\partial\mathcal{H}}{\partial x_i}(t, \boldsymbol{x}_t, \boldsymbol{u}_t^*, \boldsymbol{p}_t) - \sum_{j=1}^{n}\sum_{a=1}^{d} p_{j,t}\cdot\frac{\partial\sigma_{j,a}}{\partial x_i}dB_{a,t} \quad \text{with } p_{i,T} = \frac{\partial\Psi}{\partial x_i}; \tag{2.6}$$

(iii) the critical point condition:

$$\frac{\partial\mathcal{H}}{\partial u_c}(t, \boldsymbol{x}_t, \boldsymbol{u}_t^*, \boldsymbol{p}_t) = 0 \quad \text{for each } c \in \{1, \dots, k\}; \tag{2.7}$$

3. Optimal Trade Execution in the Almgren and Chriss Framework

We consider the problem of liquidating the amount of x_0 shares of certain stock within the time interval $[0, T]$. Let us denote by x_t the number of shares that the investor holds at the current time $t \in [0, T]$, where T denotes the liquidation process maturity. Let $\tilde{S}_t$ be the price at which we transact. In this article, we suppose that the price dynamic follows the Almgren-Chriss model. In this framework, the transacted price $\tilde{S}_t$ consists of the *unaffected* (or *fair*) price, given by a semi-martingale and denoted by S_t, and a slippage. The unaffected price S_t is the solution of the following stochastic equation:

$$\begin{cases} \text{differential form:} \quad dS_t = \gamma dx_t + \mu dt + \sigma dB_t \\ \text{integral form:} \quad S_t = S_0 - \gamma x_0 + \gamma x_t + \mu t + \sigma B_t \end{cases} \tag{3.1}$$

The mid-price, affected by trading, incorporates a slippage given by the real function h:

$$\tilde{S}_t = S_t + h(v_t), \tag{3.2}$$

where $v_t := \dot{x}_t$ (or $v_t := -\dot{x}_t$ when liquidating positions) is the *rate of trading*. In the literature, the term γdx_t is interpreted as *permanent impact*, while the function h as *temporary impact*. Following the classical Almgren-Chriss framework, we assume that $\gamma = \mu = 0$ and the temporary impact function h to be linear with respect to the rate of trading: $h(v_t) = \eta \cdot v_t$.

In this article, we consider the problem of optimal execution as a stochastic optimal control problem and we solve it using the methodology described in Section 2. With this respect, we consider the objective functional to be the *risk-unadjusted cost of trading* (*i.e.*, with no penalty for risk), given by:

$$\mathcal{C} = \mathbb{E}\left[\int_0^T \tilde{S}_t v_t \, dt\right] = \mathbb{E}\left[\int_0^T (S_t + \eta \cdot v_t) v_t \, dt\right], \tag{3.3}$$

where the state vector is $\boldsymbol{x}_t = (S_t, x_t)$, and the control variable is v_t. Thus, the control problem (2.3) becomes:

$$\max_{v_t} \left\{ -\mathbb{E}\left[\int_0^T (S_t + \eta \cdot v_t) v_t \, dt\right] \right\}, \tag{3.4}$$

The dynamics of the state variables are, respectively:

$$\begin{cases} dS_t = \sigma dB_t, \\ dx_t = -v_t dt. \end{cases} \tag{3.5}$$

The Hamiltonian stochastic functional (2.4) is, in this case:

$$\mathcal{H}(t, S_t, v_t, \boldsymbol{p}_t) = -[S_t \cdot \dot{x}_t - \eta \cdot (\dot{x}_t)^2]dt + p_{2,t} \cdot \dot{x}_t dt, \tag{3.6}$$

where $v_t := -\dot{x}_t$, and $\boldsymbol{p}_t = (p_{1,t}, p_{2,t})$ denotes the adjoint process, and all coefficients of $p_{1,t}$ are null in the expression of the Hamiltonian. It is obviously that property (2.5), applied to (3.6), generates the dynamics of the state variables, given in (3.5). The adjoint linear differential stochastic system (2.6) is reduced at one equation only (because of null coefficients of $p_{1,t}$):

$$dp_{2,t} = -[S_t \cdot \ddot{x}_t - 2 \cdot \eta \cdot \dot{x}_t \cdot \ddot{x}_t - p_{2,t} \cdot \ddot{x}_t]dt. \tag{3.7}$$

The critical point condition (2.7), applied to (3.6), yields to the representation of adjoint variable $p_{2,t}$ as function of the control variable v_t:

$$p_{2,t} = S_t + 2 \cdot \eta \cdot v_t. \tag{3.8}$$

Replacing (3.8) in (3.7), we obtain the optimal value of the control variable:

$$v_t^* = \frac{x_0}{T}. \tag{3.9}$$

With this result, we deduce the optimal trajectory of liquidation problem:

$$x_t^* = x_0 \cdot \left(1 - \frac{t}{T}\right). \tag{3.10}$$

In a similar manner, we are able to solve the general Almgren and Chriss framework, which presupposes to add the risk term, in the objective functional, that adjusts the variance of the trading cost:

$$\mathbb{VAR}[\mathcal{C}] = \mathbb{VAR}\left[\int_0^T x_t \, dS_t\right] = \sigma^2 \int_0^T x_t^2 \, dt. \tag{3.11}$$

Then, the expected cost of trading is given by:

$$\mathcal{C} = \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \cdot \sigma^2 \cdot \int_0^T x_t^2 \, dt, \tag{3.12}$$

for some price of risk λ. Interesting in this representation is the analogy to Physics: the first term is similar to kinetic energy, while the second term it seems to be the potential energy of a system. Applying the methodology exposed in Section 2, we obtain the optimal solution of this problem:

$$x_t^* = x_0 \frac{\sinh \kappa(T - t)}{\sinh \kappa T}, \text{ with } \kappa = \frac{\lambda \sigma^2}{\eta}. \tag{3.13}$$

If we consider another measure of risk (e.g., Value-at-Risk – VaR) instead of variance, the expected cost of trading is given by:

$$\mathcal{C} = \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \cdot \sigma \cdot \int_0^T x_t \, dt, \tag{3.14}$$

for some price of risk λ. Solving this problem by applying the methodology presented in Section 2, we get the optimal and state variable, respectively:

$$v_t^* = 2 \cdot x_0 \cdot \left(1 - \frac{t}{T}\right) \text{ and } x_t^* = x_0 \cdot \left(1 - \frac{t}{T}\right)^2. \tag{3.15}$$

Final Remark. All the solutions obtained in (3.9) – (3.10), (3.13) and (3.15) are concordant to those existing in the literature, so our methodology is a viable alternative to the classical one.

4. Conclusions

Alternatively to the classical literature, we proposed a new proof of the optimal schedule in the Almgren-Chriss model. This approach is different from those existing in the classical literature. As a purpose for a future research, we can formulate and solve the multi-time version of the Almgren and Chriss model. The mathematical framework was established in Udrişte and Damian (2011a). The aim of this generalization is to incorporate in price dynamics another components such as behavioral aspects in trading process.

Acknowledgements

I wish to express my heartfelt appreciation and deep sense of gratitude to my scientific advisor, prof. dr. Constantin Udrişte, who introduced me to this class of research. Also, I would like to thank in a special way to prof. dr. Bogdan Negrea (ASE Bucureşti) for his constantly encouraging support and for being a great source of inspiration and knowledge.

Note

(1) The first version of this paper was published in 2005, in *Social Science Research Network Working Paper Series*.

References

Almgren, R. and Chriss, N., 1999. Value under liquidation. *Risk.* pp. 61-63.

Almgren, R. and Chriss, N., 2000. Optimal execution of portfolio transactions. *Risk*. 3. pp. 5-39.

Avellaneda, M., Stoikov, S., 2008. High-frequency trading in a limit order book. *Quantitative Finance.* pp. 217-224.

Gatheral, J., Schied, A., 2011. Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework. *International Journal of Theoretical and Applied Finance.* 14 (3). pp. 353-368.

Guéant, O., Lehalle, C.A., Fernandez-Tapia J., 2012. Optimal portfolio liquidation with limit orders. *SIAM J. Finan. Math.* 3 (1). pp. 740-764.

Obizhaeva, A., Wang, J., 2013. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*. 16 (1). pp. 1-32.

Udrişte, C., Damian, V., 2011. Simplified single-time stochastic maximum principle. *Balkan Journal of Geometry and Its Applications.* 16 (2). pp. 155-173.

Udrişte, C., Damian, V., 2011a. Multitime stochastic maximum principle on curvilinear stochastic integral actions. ArXiv Preprint. [Online]. Available at: http://arxiv.org/abs/1112.0865 [last accesed: October 9, 2015].

Udrişte, C., 2015. Comparing variants of single-time stochastic maximum principle. In Rudas, I.J. (editor), *Recent Advances on Computational Science and Applications*. WSEAS Proceedings. pp. 23-28.