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Service Sector, Human Capital Accumulation and Endogenous Growth

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Abstract. *The present paper considers endogenous growth model with commodity sector and service sector. It is shown that there exists a unique steady state growth rate of human capital accumulation which works as the source of growth for all other sectors of the economy. This growth rate rises with the output elasticity of skilled labour in both sectors and that of physical capital in commodity sector. It is found from the comparative static analysis that increases in preference towards service good will raise growth rate only if service sector is more intensively dependent on human capital than commodity sector. It is also found that as the productivity of human capital accumulation technology increases, employment of raw labour in commodity sector increases and hence service sector becomes more skill intensive.*

Keywords: Service sector, Human Capital, Endogenous Growth.

JEL Classification: O41, O14, E24.

1. Introduction

The role of service sector is becoming increasingly important in developed as well as in developing countries. Across the globe, agriculture and manufacturing sectors have experienced phases of deceleration, stagnation and growth but the service sector has shown consistent and increasing growth. Empirical data has revealed that the service sector accounts for a significant proportion of GDP in most countries, including low income countries.(1) The process of development usually coincides with a growing role of services in the economy. Service sector makes a direct and significant contribution to GDP and job creation, and provides crucial inputs for the rest of the economy, thus having a significant effect on the national income which is an essential determinant of growth and development. The service sector produces intangible goods, such as health, education and some quite new services such as modern communications, information, and business services etc. Producing services tends to require relatively less physical capital and more human capital than producing agricultural or industrial goods. As a result demand has grown for more educated workers, prompting countries to invest more in education—an overall benefit to their people. Service sector plays a complimentary role and accelerates the process of development through quality improvement and enhancement with efficiency of productivity and developmental activities.

There exists huge substantial empirical literature dealing with service sector. Empirical literature includes the works of Fiala (1983) and (1980), Chand (1983), Lorence (1991), Lee and Wolpin (2006), Kirn (1987), Wood (1990), Marks (2006).

Eswaran and Kotwal (2002), Sasaki (2007), Konan and Maskus (2006), Baumol et al. (1989), Pugno (2006), Zhang (2013) have contributed to the theoretical works of service sector. But none of these studies considered the issue of human capital accumulation and endogenous growth model with service sector, which is the main focus of the present analysis. Uzawa (1963)(2) model has considered a perishable consumption good sector that has the same feature like the service good in our model. However, Uzawa (1963) model did not consider the case of human capital accumulation and basically it was an exogenous growth model with zero rate of growth.

The present paper constructs an endogenous growth model with human capital accumulation and service sector. The study focuses on the role of service sector on economic growth and tries to find effects of different parameters on growth rate using comparative static analysis for further policy prescriptions. This paper is organised as follows: In section 2, the basic model is presented, section 3 describes the growth rates along with the steady state, comparative static analysis is done in section 4, section 5 concludes.

2. The model

This paper considers a closed economy model with two sectors namely, commodity sector and service sector. The total labour force is heterogeneous with respect to skill level: skilled labour and unskilled labour or raw labour. The commodity and the factor markets are characterized by perfect competition. The economy is inhabited by identical rational agents. Production technology is subject to constant returns to scale. Preferences over consumption and service streams are given by the following function where 'c' and 's' denote flow of real per capita consumption of commodity and service respectively:

$$U(c,s)=\frac{(c^{\alpha}s^{1-\alpha})^{1-\sigma}-1}{(1-\sigma)} \tag{1}$$

Here, we assume that the output in the commodity sector can be used for consumption or investment. However, the output in the service sector is fully consumed as services cannot be stored for future uses. The commodity output is a function of human capital, raw labor and physical capital. The services are produced with human capital and raw labor only. Both the production functions are Cobb-Douglas type. Skilled labour allocates 'u' fraction of time to produce commodity output, 'v' fraction of time to produce service output and (1-u-v) fraction of time for human capital accumulation. Let 'K' be the level of physical capital in commodity sector. Let α_1 and β be the service output elasticity and commodity output elasticity of skilled labour respectively and α_2 be the commodity output elasticity of physical capital. The service and commodity output production functions are as follows where 'y_s' and 'y_c' be the flow of commodity output and service output respectively.

$$y_s=(N_1vh)^{\beta}((1-\phi)N_2)^{1-\beta} \tag{2}$$

$$y_c=(N_1uh)^{\alpha_1}K^{\alpha_2}(\phi N_2)^{1-\alpha_1-\alpha_2} \tag{3}$$

It is assumed that the number of skilled labor and unskilled labor be N_1 and N_2 respectively and N be the total labour force or working population such that

$$N=N_1+N_2 \tag{4}$$

Population grows at a constant, exogenous rate and more over we assume that each segment of the population is growing at the same rate in the following way:

$$\frac{\dot{N}}{N}=\frac{\dot{N}_1}{N_1}=\frac{\dot{N}_2}{N_2}=n\geq 0 \tag{5}$$

It is further assumed that the general skill level of a worker is 'h'. The effective skilled work force in commodity production is 'N_1uh' and that in service production is 'N_1vh'. These are skill-weighted man-hours devoted to current commodity and service output production respectively. Let 'ϕ' be the fraction of total unskilled labor force that is devoted in the commodity sector. The remaining (1- ϕ) fraction is engaged in producing service output.

Following Lucas (1988), the accumulation of human capital is assumed to be proportional to the time allocated for education. Hence, human capital accumulation function is given by

$$\gamma_h = \dot{h}/h = \delta(1-u-v) \tag{6}$$

Here δ be the productivity parameter in the human capital accumulation function. It is assumed that commodity output over aggregate consumption is accumulated as physical capital. The physical capital accumulation function is given by

$$\dot{K} = y_c - Nc \tag{7}$$

Service output is totally consumed. So the market clearing condition is

$$y_s = Ns \tag{8}$$

The objective of the economy is to maximize the value of utility defined by equation (1) subject to the constraints given by (2), (3), (6), (7) and (8).

The current value Hamiltonian of this dynamic optimization problem can be formulated as

$$H = \frac{N}{(1-\sigma)}[(c^{\alpha}s^{1-\alpha})^{1-\sigma}-1] + \theta_1[(N_1uh)^{\alpha_1}K^{\alpha_2}(\phi N_2)^{1-\alpha_1-\alpha_2} - Nc]$$
$$+\theta_2[\delta h(1-u-v)] + \lambda[(N_1vh)^{\beta}((1-\phi)N_2)^{1-\beta} - Ns] \tag{9}$$

Where θ_1 and θ_2 are shadow prices associated with $\dot{K}$ and $\dot{h}$ respectively and λ be the Lagrange multiplier as the market clearing condition given by equation (9) is a static constraint. There are five control variables c, s, u, v, ϕ in this model. The state variables are K and h (Please see Appendix for details derivation).

3 Steady state Growth path

The model results show that along the steady state growth path c, s , K grow at constant rate and u, v, φ are time independent. In the steady state the growth path of human capital is given in equation (10a).

$$\gamma_h = \delta(1-u-v) = \frac{A}{B} \tag{10a}$$

Here $A = n + \delta - \rho$ (10b)

$$B = 1 - x \tag{10c}$$

Where $x = (1-\sigma)\{\beta(1-\alpha) + \frac{\alpha\alpha_1}{(1-\alpha_2)}\}$ (10d)

Let γ_i denotes the growth rate of i^{th} variable where i= c, s, and K. Equilibrium growth path for these variables are given as follows:

$$\gamma_c = \left(\frac{\alpha_1}{1-\alpha_2}\right)\gamma_h \tag{11a}$$

$$\gamma_s = \beta\gamma_h \tag{11b}$$

$$n + \gamma_c = \gamma_K \tag{11c}$$

We observe that growth rate of human capital is the source of growth for other three sectors. Thus, this model with service sector where entire service output is consumed confirms the findings of Lucas model that human capital accumulation is the engine of growth.

From the first order conditions of optimization we obtain the values of u, v and ϕ defined as follows:

$$u = (\delta B - A)/\delta B(1+D) \tag{12a}$$

$$v = D(\delta B - A)/\delta B(1+D) \tag{12b}$$

And $\phi = \dfrac{(1-\alpha_1-\alpha_2)\beta}{\alpha_1(1-\beta)D+(1-\alpha_1-\alpha_2)\beta}$ (12c)

Where D is a constant and given by

$$D = \frac{(1-\alpha)\beta}{\alpha\alpha_1}\left[\frac{(\rho - n\alpha_2)Z - (n+\delta-\rho)(Y+\alpha_1\alpha_2)}{\rho Z - (n+\delta-\rho)Y}\right] \tag{12d}$$

Where $Z = (1-\alpha_2)[1-(1-\sigma)\{\beta(1-\alpha)+\dfrac{\alpha\alpha_1(1-\sigma)}{(1-\alpha_2)}\}]$ (12e)

And $Y = \alpha_1\{\alpha(1-\sigma)-1\}+\beta(1-\alpha)(1-\alpha_2)(1-\sigma)$ (12f)

Hence in this model the values of u, v and ϕ are endogenously determined. The model to be economically viable u, v and 1-u-v should be positive.

The conditions for positive u, v and (1-u-v) are:

When $\sigma<1$; the positivity of u and v requires the following condition:

$\delta B \geq A$

Or, $\delta(1-x) \geq n+\delta-\rho$

Or, $\rho \geq n+\delta x$ (13a)

From $(1-u-v)\geq 0$, we get $\dfrac{A}{\delta B}\geq 0$

Therefore the required conditions are:

A and B must be positive. So, from the requirement that $A \geq 0$ and $B \geq 0$ we get (13b) and (13c) equations respectively

$n+\delta \geq \rho$ (13b)

And $1-x \geq 0$,

Or. $x \leq 1$ (13c)

Merging equation no. 13a.and 13b we get,

$n+\delta \geq \rho \geq n+\delta x$

This is the same condition that is required for positive B. Combining extreme left and extreme right side of the above inequality we in fact have (13c). When $\sigma \geq 1$: The required conditions are the same as the previous case.

Now the positivity of D is also required for positive values of u and v.

The conditions required for positive D:

When $\sigma<1$; the sufficient condition $\alpha_1 > \alpha_1^*$

Where $\alpha_1^* = \dfrac{\alpha_2 + (1-\sigma)\beta(1-\alpha)(1-\alpha_2)}{1-(1-\sigma)\alpha}$.

Thus, combining the conditions we have the Lemma 1a follows:

Lemma 1. *There exists positive, unique steady state growth rate for human capital consumption service and capital goods sectors.*

A special case: $\sigma = 1$

When $\sigma = 1$ the utility function can be represented as $\alpha \ln c + (1-\alpha)\ln s$

From (10c), we have B = 1.

Further, from (12e) and (12f) we have the following results:

$Z = (1-\alpha_2)$

$Y = -\alpha_1$

Given these values for Z and Y, 12d confirms that D>0 for $\rho \geq n\alpha_2$

Hence, with $\sigma = 1$ from (10a) we get

$\gamma_h = \dfrac{n+\delta-\rho}{\delta} > 0$ for $n+\delta \geq \rho$.

Further, as u + v < 1, we must have. This in turn confirms that D > 0 as for. Thus, we observe that when growth rate of human capital does not depend on preference parameters and output of different factors of production.

Further, 11a, 11b and 11c show that growth rate of human capital is the source of growth for other three sectors. Hence, we also have $\gamma_c > 0, \gamma_s > 0, \gamma_K > 0$ and also, given the values of Z and Y, as D > 0 we have u > 0, v > 0 and Φ > 0. Thus our model depicts positive steady state growth rate for all variables.

4 Comparative Static Analysis

Since human capital growth rate is the driving force of commodity and service growth rate. In this section, we do the comparative static analysis of γ_h with respect to some parameters of this model namely α, β, α_1, α_2. We find that

$$\frac{d\gamma_h}{d\alpha} = \frac{(1-\sigma)(\frac{\alpha_1}{1-\alpha_2} - \beta)(n+\delta-\rho)}{[1-x]^2} \tag{14a}$$

Where x is given by equation 10d.

Increase in α raises human capital growth rate if $\frac{\alpha_1}{1-\alpha_2} > \beta$. This condition implies that if the ratio of output elasticity of skilled labour in commodity sector and output elasticity of total labour (skilled and unskilled) force employed in the commodity sector is higher than output elasticity of human capital in service sector, increase in preference towards commodity consumption will raise growth rate of human capital and hence overall growth rates and vice versa. In other words, preference towards service sector will raise growth rate only if output elasticity of human capital in service sector is higher than the ratio of the output elasticity of human capital in commodity sector and the output elasticity of total labour force employed in the commodity sector. The logic lies in the fact that as preference towards service (consumption) increases, overall growth rate will rise if service sector (consumption) is more intensively dependent on human capital than commodity (service) sector.

$$\frac{d\gamma_h}{d\beta} = \frac{(1-\sigma)(1-\alpha)(n+\delta-\rho)}{[1-x]^2} \tag{14b}$$

$$\frac{d\gamma_h}{d\alpha_1} = \frac{(1-\sigma)(\frac{\alpha_1}{1-\alpha_2})(n+\delta-\rho)}{[1-x]^2} \tag{14c}$$

Where x is given by equation (10d).

The increase in output elasticity of skilled labour in commodity sector as well as service sector raises the growth rate of human capital. This is because increase in output elasticity of skilled labour increases the marginal productivity of skilled labour. This leads to an increase in real wage of skilled labours. In this dynamic model, hike in wage gives an incentive to the labours to devote more time to accumulate human capital. Hence the growth rate of human capital increases.

$$\frac{d\gamma_h}{d\alpha_2} = \frac{\alpha\alpha_1(1-\sigma)(n+\delta-\rho)}{[1-x]^2(1-\alpha_2)^2} \tag{14d}$$

The marginal productivity of physical capital increases with the increase in output elasticity of physical capital in commodity sector. This raises the return to physical capital. So individuals get incentive to invest more in physical capital and this needs more production of commodity sector because the commodity output over consumption is invested as physical capital. Since commodity output production is dependent on skilled labour and human capital growth drives the commodity production growth, more commodity production needs more human capital investment. This gives the workers an incentive to devote more time to accumulate human capital.

Proposition 1*:* *Preference towards service sector will raise growth rate only if output elasticity of human capital in service sector is higher than the ratio of output elasticity of skilled labour in commodity sector and the output elasticity of total labour force employed in the commodity sector.*

Like the Lucas (1988) model in this paper also human capital is the engine of growth. So as preference towards service (consumption) increases, overall growth rate will rise if service sector (consumption) is more intensively dependent on human capital than commodity (service) sector.

Proposition 2: *If σ < 1, increase in output elasticity of human capital in service sector or/and commodity sector or/and increase in output elasticity of physical capital in commodity sector will raise growth rate. When σ = 1, growth rate of human capital neither depends on output elasticity of any factor of production nor marginal utility of any argument of utility function.*

Now we evaluate the effect of change in productivity parameter in the human capital accumulation function on employment share. Here ϕ is the employment share of raw labour in commodity sector.

We have found that (See the appendix for the derivation in detail):

$\frac{\partial\phi}{\partial\delta} > 0$ and when $\frac{\partial u}{\partial\delta}$ decreases, $\frac{\partial v}{\partial\delta}$ also decreases as $v = Du$ and $\frac{\partial D}{\partial\delta} < 0$

Which implies that as the productivity parameter in the human capital accumulation sector works more efficiently the employment share of the unskilled labour increases in the commodity sector. If the number of hours dedicated for commodity production sector decreases due to rise in productivity parameter in human capital accumulation sector, the number of hours devoted for service good production also decreases. As a result of increase in δ and decrease in the values of u and v the growth rate of human capital accumulation function $\gamma_h = \delta(1-u-v)$ becomes higher. As δ rises the productivity of skilled labour increases. Therefore the wage rises for the skilled labour. As a result the consumption demand for commodity output and service output in both sectors increases. The growth rate of physical capital also increases for having more productive skilled labour. The productivity of raw labour rises due to more productive human capital and physical capital to work with. So there is also a hike in wage rate in unskilled labour market. As employment of human capital and physical capital both are increased in the commodity output production sector, marginal productivity of raw labour in commodity sector increases and hence more raw labours flow to commodity sector.

Proposition 3: *As the productivity of human capital accumulation technology increases, employment of raw labour in commodity sector increases and hence service sector becomes more skill intensive.*

5. Conclusion

Existing endogenous growth models dealing with human capital accumulation have not considered service sector as one of the most important sectors. Generally, the endogenous growth models deal with commodity production only. The growth of service sector has not been taken into account in such endogenous growth models where human capital acts as a driving force for the growth of the whole economy. On the other hand, none of the service sector models are endogenous growth models. This paper tries to fill the gap by developing an endogenous growth model with commodity sector and service sector. It is shown that there exists unique steady state growth rate of human capital accumulation which works as the main driving force for the economy as a whole rises with the output elasticity of skilled labour in both sectors and that of physical capital in commodity sector. It is found from the comparative static analysis that increases in preference towards service sector will raise growth rate only if service sector is more intensively dependent on human capital than commodity sector.

Notes

(1) The service sector has the highest sectoral contribution in global GDP with a share of 67.5 per cent in world GDP of US$70.2 trillion in 2011, as per UN National Account Statistics. In the last two decades Service sector has provided more than 60 per cent of global output and, in many countries, an even larger share of employment. Service sector has been the main economic activity and source of employment in the world economy for decades. It accounts for around 2/3rd of the world output and slightly less than half of the world employment in 2011 (Sources: UNCTAD Stat, ILO Global Employment Trends 2012).Employment shares of services have shown similar trend as output trends. As the sector's global output reaches up to $40 trillion, it provides employment for 1.4 billion people in the world. Significant differences between developed and developing countries exist. The sector occupies significantly larger share in developed economies compared with developing countries. The services sector is gradually becoming the world's main sector for employment (Source: ILO Global Employment Trends 2012).

(2) Uzawa (1963) has worked on two sector growth model, considering an economy in which there exist two productive sectors, one producing consumption goods and other producing capital goods. Consumption goods are instantaneously consumed, while capital goods depreciate at a fixed rate and a constant fraction is saved out of the current gross national product.

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Appendix

The current value Hamiltonian can be formulated as

$$H = \frac{N}{(1-\sigma)}[(c^{\alpha} s^{1-\alpha})^{1-\sigma} - 1] + \theta_1[(N_1 uh)^{\alpha_1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} - Nc]$$

$$+\theta_2[\delta h(1-u-v)] + \lambda[(N_1 vh)^{\beta}((1-\phi)N_2)^{1-\beta} - Ns] \qquad \text{(A.1)}$$

There are five decision variables in this problem. The decision variables are c, s, u, v, ϕ.

The first order conditions are

$dH \,/\, dc = 0$

Or, $\frac{N}{(1-\sigma)}(1-\sigma)\alpha c^{\alpha(1-\sigma)-1} s^{(1-\alpha)(1-\sigma)} - N\theta_1 = 0$

$$\text{Or } \alpha c^{\alpha(1-\sigma)-1} s^{(1-\alpha)(1-\sigma)} = \theta_1 \qquad \text{(A.2)}$$

Taking logarithm both sides and differentiating w.r.t time

$$\{\alpha(1-\sigma)-1\}\dot{c}/c + (1-\sigma)(1-\alpha)\dot{s}/s = \dot{\theta}_1/\theta_1 \qquad \text{(A.3)}$$

$dH \,/\, ds = 0$

$\frac{N}{(1-\sigma)}(1-\alpha)(1-\sigma)c^{\alpha(1-\sigma)} s^{(1-\alpha)(1-\sigma)-1} - \lambda N = 0$

$$\text{Or, } (1-\alpha)c^{\alpha(1-\sigma)} s^{(1-\alpha)(1-\sigma)-1} = \lambda \qquad \text{(A.4)}$$

$dH \,/\, du = 0$

$$\text{Or, } \theta_1 \alpha_1 u^{\alpha_1 - 1}(N_1 h)^{\alpha_1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} - \theta_2 \delta h = 0 \qquad \text{(A.5)}$$

Taking logarithm both sides and differentiating with respect to time

$$\dot{\theta}_1/\theta_1 + (\alpha_1 - 1)\delta(1-u-v) + (1-\alpha_2)n + \alpha_2 \gamma_k = \dot{\theta}_2/\theta_2 \qquad \text{(A.6)}$$

$dH/dv = 0$

$$\text{Or, } \theta_2 \delta h = \lambda \beta v^{\beta-1}(N_1 h)^{\beta}((1-\phi)N_2)^{1-\beta} \qquad \text{(A.7)}$$

Two co-state equations are

$$\dot{\theta}_1 = \rho\theta_1 - dH/dK \qquad \text{(A.8)}$$

$$\dot{\theta}_2 = \rho\theta_2 - dH/dh \tag{A.9}$$

Now $dH/dK = \theta_1 (N_1 uh)^{\alpha_1} \alpha_2 K^{\alpha_2 - 1} (\phi N_2)^{1-\alpha_1-\alpha_2}$ (A.10)

Substituting this value into equation (A.8) we get

$$\dot{\theta}_1 = \rho\theta_1 - \theta_1 (N_1 uh)^{\alpha_1} \alpha_2 K^{\alpha_2 - 1} (\phi N_2)^{1-\alpha_1-\alpha_2}$$

Or $\dot{\theta}_1 / \theta_1 = \rho - (N_1 uh)^{\alpha_1} \alpha_2 K^{\alpha_2 - 1} (\phi N_2)^{1-\alpha_1-\alpha_2}$ (A.11)

Now

$$dH/dh = \theta_1 (N_1 u)^{\alpha_1} \alpha_1 h^{\alpha_1 - 1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} + \theta_2 \delta (1-u-v) + \lambda\beta h^{\beta-1} (N_1 v)^{\beta} ((1-\phi)N_2)^{1-\beta} \tag{A.12}$$

Substituting this value into equation (A.8) we get

$$\dot{\theta}_2 = \rho\theta_2 - \theta_1 (N_1 u)^{\alpha_1} \alpha_1 h^{\alpha_1 - 1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} - \theta_2 \delta (1-u-v) - - \lambda (N_1 v)^{\beta} \beta h^{\beta-1} ((1-\phi)N_2)^{1-\beta} \tag{A.13}$$

Dividing both sides by θ_2 and substituting the values of θ_1/θ_2 and λ/θ_2 from equations (A.5.) and (A.7) respectively we get

$$\dot{\theta}_2 / \theta_2 = \rho - \delta \tag{A.14}$$

$$dH/d\phi = 0$$

$$\theta_1 (1-\alpha_1-\alpha_2)(N_1 uh)^{\alpha_1} K^{\alpha_2} N_2^{1-\alpha_1-\alpha_2} \phi^{-(\alpha_1+\alpha_2)} = \lambda (N_1 vh)^{\beta} (1-\beta)(1-\phi)^{-\beta} N_2^{1-\beta} \tag{A.15}$$

Equating the values of $\theta_2 \delta h$ from the equations (A.5) and (A.7)

$$\theta_1 \alpha_1 u^{\alpha_1 - 1} (N_1 h)^{\alpha_1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} = \lambda\beta v^{\beta-1} (N_1 h)^{\beta} ((1-\phi)N_2)^{1-\beta} \tag{A.16}$$

Substituting the values of θ_1 and λ from equations (A.2) and (A.4) into the above equation and simplifying the calculation we get

$$s/c = (1-\alpha)\beta v^{\beta-1} (N_1 h)^{\beta} ((1-\phi)N_2)^{1-\beta} / \alpha\alpha_1 u^{\alpha_1 - 1} (N_1 h)^{\alpha_1} K^{\alpha_2} (\phi N_2)^{1-\alpha_1-\alpha_2} \tag{A.17}$$

Now replacing the values of θ_1 and λ from (A.2) and (A.4) into equation (A.15.22)

$$s/c = (1-\alpha)(N_1 vh)^{\beta} (1-\beta)(1-\phi)^{-\beta} N_2^{1-\beta} / \alpha (N_1 uh)^{\alpha_1} K^{\alpha_2} N_2^{1-\alpha_1-\alpha_2} (1-\alpha_1-\alpha_2) \phi^{-(\alpha_1+\alpha_2)} \tag{A.18}$$

Equating the expressions of s/c from (A.17) and (A.18) and we get

$$(\frac{v}{u}).\frac{(1-\beta)}{(1-\alpha_1-\alpha_2)}=\frac{\beta(1-\phi)}{\alpha_1\phi}$$

Or, $$(\frac{v}{u})=\frac{\beta(1-\alpha_1-\alpha_2)(1-\phi)}{\alpha_1\phi(1-\beta)} \tag{A.19}$$

Taking logarithm both sides of equation (A.17) and differentiating with respect to time we get

$$\dot{s}/s-\dot{c}/c=\alpha_2 n+(\beta-\alpha_1)\delta(1-u-v)-\alpha_2\gamma_K \tag{A.20}$$

Now, $y_s=Ns$ (by assumption)

So, $$(N_1vh)^\beta((1-\phi)N_2)^{1-\beta}=Ns \tag{A.21}$$

Taking logarithm both sides of equation (A.28) and differentiating with respect to time

$$\dot{s}/s=\gamma_s=\beta\delta(1-u-v) \tag{A.22}$$

Substituting the value of $\dot{s}/s$ from (A.22) into (A.20) we get

$$\dot{c}/c=\alpha_2\gamma_k+\delta\alpha_1(1-u-v)-\alpha_2 n \tag{A.23}$$

Or, $$\gamma_c=\alpha_2\gamma_k+\delta\alpha_1(1-u-v)-\alpha_2 n \tag{A.24}$$

Now $\dot{K}=Y-Nc$

Or $$\dot{K}/K=\gamma_K=y_c/K-Nc/K \tag{A.25}$$

Following the same logic as Lucas model we use the result $n+\gamma_c=\gamma_K$ (A.26)

Replacing the value of γ_K from (A.26) into (A.24) and solving the expression for γ_c we have

$$\gamma_c=\frac{\alpha_1}{(1-\alpha_2)}\delta(1-u-v) \tag{A.27}$$

Substituting the value of λ from (A.4) into (A.7) we rewrite the equation

$$\theta_2\delta h=c^{\alpha(1-\sigma)}(1-\alpha)s^{(1-\alpha)(1-\sigma)-1}\beta v^{\beta-1}(N_1h)^\beta\{(1-\phi)N_2\}^{1-\beta} \tag{A.28}$$

Taking logarithm both sides of equation (A.28) and differentiating w.r.t time we get

$$\begin{aligned}\dot{\theta}_2/\theta_2+\dot{h}/h&=\alpha(1-\sigma)\gamma_c+\{(1-\alpha)(1-\sigma)-1\}\gamma_s+\beta(\dot{N}_1/N_1)+\\&+\beta\dot{h}/h+(1-\beta)\dot{N}_2/N_2\end{aligned} \tag{A.29}$$

Previously we got $\dot{\theta}_2 / \theta_2 = \rho - \delta$.

Substituting this value and using the expressions of γ_c and γ_s in equation (A.29.36) we get

$$\rho - \delta = \delta(1-u-v)[\beta(1-\alpha)(1-\sigma) - 1 + \frac{\alpha\alpha_1(1-\sigma)}{(1-\alpha_2)}] + n \qquad \text{(A.30)}$$

Or, $$(u+v) = 1 - \frac{n+\delta-\rho}{\delta[1-(1-\sigma)\{\beta(1-\alpha) + \frac{\alpha\alpha_1}{(1-\alpha_2)}\}]} \qquad \text{(A.31)}$$

Or, $(u+v) = 1 - (\frac{A}{\delta B})$ (A.32)

Here $A = n + \delta - \rho$

$$B = [1-(1-\sigma)\{\beta(1-\alpha) + \frac{\alpha\alpha_1}{(1-\alpha_2)}\}]$$

Using the market clearing condition and substituting the value of λ we get,

$$\frac{dH/dc}{dH/dv} = \frac{\theta_1}{\theta_2\delta h} = \frac{\alpha v}{(1-\alpha)\beta Nc}$$

Or, $\theta_2\delta h = \dfrac{\theta_1(1-\alpha)\beta(Nc)}{\alpha v}$

From equation (A.5)

$$\theta_2\delta h = \theta_1\alpha_1 u^{\alpha_1-1}(N_1 h)^{\alpha_1} K^{\alpha_2}(\phi N_2)^{1-\alpha_1-\alpha_2}$$

Using the commodity production function in the above expression we get

$$\theta_2\delta h = \frac{\theta_1\alpha_1 y_c}{u}$$

Equating the expressions of $\theta_2\delta h$ from the above two equations we get

Or, $u/v = \dfrac{\alpha\alpha_1}{\beta(1-\alpha)}\{\dfrac{(y_c/K)}{(Nc/K)}\}$ (A.33)

Rewriting (A.11)

$$\dot{\theta}_1/\theta_1 = \rho - (N_1 uh)^{\alpha_1}\alpha_2 K^{\alpha_2-1}(\phi N_2)^{1-\alpha_1-\alpha_2} \qquad \text{(A.34)}$$

Or, $\dot{\theta}_1/\theta_1 = \rho - \alpha_2\{(N_1uh)^{\alpha_1}K^{\alpha_2}(\phi N_2)^{1-\alpha_1-\alpha_2}\}/K$

Or, $\dot{\theta}_1/\theta_1 = \rho - \alpha_2(y_c/K)$ [From the commodity production function]

Or, $(y_c/K) = [\rho - \dot{\theta}_1/\theta_1]/\alpha_2$

Or, $(y_c/K) = [(\rho/\alpha_2) - (\frac{1}{\alpha_2})(\frac{\dot{\theta}_1}{\theta_1})]$ (A.35)

Rewriting equation (A.3)

$\{\alpha(1-\sigma)-1\}\dot{c}/c + (1-\sigma)(1-\alpha)\dot{s}/s = \dot{\theta}_1/\theta_1$

Substituting this value of $\dot{\theta}_1/\theta_1$ into equation (A.35) we get

$$(y_c/K) = [(\rho/\alpha_2)] - [\frac{\{\alpha(1-\sigma)-1\}}{\alpha_2}]\gamma_c - \frac{[(1-\sigma)(1-\alpha)]}{\alpha_2}]\gamma_s$$

Substituting the values of γ_c and γ_s from equation (A.24) and (A.22) respectively and after simplification we get

$$(y_c/K) = (\frac{\rho}{\alpha_2}) - \frac{1}{\alpha_2}[\{\alpha(1-\sigma)-1\}\frac{\alpha_1}{(1-\alpha_2)}\gamma_h + (1-\alpha)(1-\sigma)\beta\gamma_h]$$

We know

$\dot{K}/K = \gamma_K = y_c/K - Nc/K$

Again following the logic from the Lucas model we get

$n + \gamma_c = \gamma_K$

So, $n + \gamma_c = (y_c/K) - (Nc/K)$

Or, $(Nc/K) = (y_c/K) - n - \gamma_c$

Or $(y_c/K) = (Nc/K) + n + \gamma_c$

$$(\frac{Nc}{K}) = (\frac{\rho}{\alpha_2}) - \frac{1}{\alpha_2}[\{\alpha(1-\sigma)-1\}\frac{\alpha_1}{(1-\alpha_2)}\gamma_h + (1-\alpha)(1-\sigma)\beta\gamma_h] - n - \gamma_c$$

From (A.33) we can write

$$\frac{v}{u} = \frac{\beta(1-\alpha)}{\alpha_1\alpha}\{\frac{(Nc/K)}{(y_c/K)}\} = \frac{(Nc/K)}{(Nc/K)+n+\gamma_c}$$

Or, $\frac{v}{u}$ = D (Say)

So we get three equations and three unknowns. Three equations are

$$(\frac{v}{u}) = \frac{\beta(1-\alpha_1-\alpha_2)(1-\phi)}{\alpha_1\phi(1-\beta)} \qquad \text{(i)}$$

$$u+v = 1-\frac{A}{\delta B} \qquad \text{(ii)}$$

$$\frac{v}{u} = D \qquad \text{(iii)}$$

Solving the values we get

$$\phi = \frac{\beta(1-\alpha_1-\alpha_2)}{\alpha_1(1-\beta)D+\beta(1-\alpha_1-\alpha_2)}$$

$$u = \frac{(\delta B - A)}{\delta B(1+D)}$$

$$v = \frac{(\delta B - A)D}{\delta B(1+D)}$$

Here we derive the required condition for positivity of D.

$$D = \frac{\{(1-\alpha)\beta\}}{\alpha\alpha_1}\frac{)(\frac{Nc}{K})}{(\frac{y_c}{K})} \qquad (1)$$

Now, $$\frac{y_c}{K} = \frac{\rho}{\alpha_2} - \{\frac{\alpha(1-\sigma)-1}{\alpha_2}\}\gamma_c - \frac{(1-\alpha)(1-\sigma)}{\alpha_2}\gamma_s \qquad (2)$$

$$\frac{Nc}{K} = \frac{\rho}{\alpha_2} - \{\frac{\alpha(1-\sigma)-1}{\alpha_2}\}\gamma_c - \frac{(1-\alpha)(1-\sigma)}{\alpha_2}\gamma_s - n - \gamma_c \qquad (3)$$

So, $\frac{Nc}{K} = \frac{y_c}{K} - n - \gamma_c$

So, $D = \frac{\{(1-\alpha)\beta\}}{\alpha\alpha_1}\frac{(\frac{Nc}{K})}{(\frac{y_c}{K})} = \frac{\frac{Nc}{K}}{\frac{Nc}{K}+n+\gamma_c}$

So for D to be positive, $\frac{Nc}{K}$ should be positive and if $\frac{Nc}{K}$ is positive, $\frac{y_c}{K}$ must be positive (it is clear from the above relation).

Now, $\frac{Nc}{K} = \frac{\rho}{\alpha_2} - \{\frac{\alpha(1-\sigma)-1}{\alpha_2}\}\gamma_c - \frac{(1-\alpha)(1-\sigma)}{\alpha_2}\gamma_s - n - \gamma_c > 0$

Here $\gamma_c = (\frac{\alpha_1}{1-\alpha_2})\gamma_h$ and $\gamma_c = \beta\gamma_h$

Substituting these values we have

$$\frac{Nc}{K} = \frac{\rho}{\alpha_2} - \frac{1}{\alpha_2}[\{\alpha(1-\sigma)-1\}(\frac{\alpha_1}{1-\alpha_2})\gamma_h + (1-\alpha)(1-\sigma)\beta\gamma_h + \frac{\alpha_1\alpha_2}{1-\alpha_2}\gamma_h]$$
$$-n$$

Or, $\frac{Nc}{K} = \frac{1}{\alpha_2}[\rho - \{(\alpha(1-\sigma)-1+\alpha_2)(\frac{\alpha_1}{1-\alpha_2})\}\gamma_h - (1-\alpha)(1-\sigma)\beta\gamma_h - n]$

$$\frac{Nc}{K} = \frac{1}{\alpha_2}[\rho - \gamma_h\{(1-\alpha)(1-\sigma)\beta + (\frac{\alpha_1}{1-\alpha_2})\{\alpha(1-\sigma)+\alpha_2-1\}\} - \alpha_2 n]$$

Or, $\frac{Nc}{K} = \frac{1}{\alpha_2}[\rho + \gamma_h\{\alpha_1 - (1-\sigma)\{\beta(1-\alpha) + \frac{\alpha_1\alpha}{1-\alpha_2}\}\} - \alpha_2 n]$

Here $x =$

$$\frac{Nc}{K} = \frac{1}{\alpha_2}[\rho + \frac{(n+\delta-\rho)}{(1-x)}(\alpha_1 - x) - \alpha_2 n]$$

$$\frac{Nc}{K} = \frac{1}{\alpha_2}[\frac{(1-\alpha_1)}{(1-x)}\rho + n\{\frac{(\alpha_1-x)}{(1-x)} - \alpha_2\} + \frac{\delta(\alpha_1-x)}{(1-x)}]$$

Here $x = (1-\sigma)\{\beta(1-\alpha) + \frac{\alpha\alpha_1}{(1-\alpha_2)}$

$L.H.S \geq 0$

Or $\alpha_1 \geq x$

Or, $\alpha_1 - x \geq \alpha_2(1-x)$

Or, $\alpha_1 \geq x + \alpha_2(1-x)$

Substituting the value of x in the above inequality we get

$\alpha_1 \geq \alpha_2 + (1-\sigma)\{\beta(1-\alpha)(1-\alpha_2) + \alpha\alpha_1\}$

Or, $\alpha_1[1-(1-\sigma)\alpha] \ \geq \alpha_2 + (1-\sigma)\beta(1-\alpha)(1-\alpha_2)$

Or, $\alpha_1 \geq \dfrac{\alpha_2 + (1-\sigma)\beta(1-\alpha)(1-\alpha_2)}{1-(1-\sigma)\alpha} = \alpha_1^*$

$\alpha_1^* < 1$

Or, $1 > \alpha_2 + \alpha(1-\sigma) + \beta(1-\alpha)(1-\sigma)(1-\alpha_2)$

Now we rewrite the expression of D

$$D = \frac{(1-\alpha)\beta}{\alpha\alpha_1}\left[\frac{(\rho - n\alpha_2)Z - (n+\delta-\rho)(Y+\alpha_1\alpha_2)}{\rho Z - (n+\delta-\rho)Y}\right]$$

Or $D = \dfrac{(1-\alpha)\beta}{\alpha\alpha_1} - \dfrac{\{n\alpha_2 z + \alpha_1\alpha_2(n+\delta-\rho)\}}{\rho Z - (n+\delta-\rho)Y}\dfrac{(1-\alpha)\beta}{\alpha\alpha_1}$

Where

$$Z = (1-\alpha_2)[1-(1-\sigma)\{\beta(1-\alpha) + \frac{\alpha\alpha_1(1-\sigma)}{(1-\alpha_2)}\}$$

And $Y = \alpha_1\{\alpha(1-\sigma)-1\} + \beta(1-\alpha)(1-\alpha_2)(1-\sigma)$

Now, $\dfrac{\partial D}{\partial \delta} = -\dfrac{(1-\alpha)\beta}{\alpha\alpha_1}\left[\dfrac{\rho z\alpha_1\alpha_2 + n\alpha_2 Z}{\{\rho Z - (n+\delta-\rho)Y\}}\right] < 0$

And we have found previously

v=Du

Differentiating both sides by δ we have

$$\frac{\partial v}{\partial \delta} = u\frac{\partial D}{\partial \delta} + D\frac{\partial u}{\partial \delta}$$

From the above equation it is clear that if $\dfrac{\partial u}{\partial \delta} < 0$, $\dfrac{\partial v}{\partial \delta}$ will be negative for sure as $\dfrac{\partial D}{\partial \delta}$ is negative.