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Efficiency, non-linearity and chaos: evidences from BRICS foreign exchange markets

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Abstract. *This study examines the presence of possible weak form of market efficiency, non-linearity and chaotic behaviour in the foreign exchange markets of Brazil, Russia, India, China and South Africa(commonly known together as BRICS) using various tests. Monthly Nominal Effective Exchange Rate (NEER) data for the said countries, ranging from April 1994 to September 2014 were examined. The analysis was carried out using variance ratio (VR) tests, BDS test, Hinich Bispectrum test, Teräsvirta Neural Network test and estimation of Largest Lyapunov exponents (LLE's). The first stage consisted of testing for weak-form market efficiency using VR tests, which rejected the Weak-form Efficient Market Hypothesis for all the five series. The next step was to analyze the possible presence of non-linearity, and this was carried out using BDS test, Hinich Bispectrum test, and Teräsvirta Neural Network test. All the tests confirmed the presence of non-linearity in the five series under study. The last step of the study was to analyze the structure of non-linearity. LLE's were estimated for all series in order to examine the possible presence of chaos. LLE's for all the 5 series returned positive values, and thus confirmed the presence of an underlying chaotic structure for all markets.*

Keywords: Forex Market, BRICS, India, Chaos, Efficiency.

JEL Classification: G14, G15, F31.

Introduction

Behaviour of financial markets is an area of interest to researchers, investors and policy makers alike. Among the hypotheses proposed to explain market behaviour, efficient market hypothesis is the most discussed one.

The efficient market hypothesis in its weak form (Fama, 1965) states that a security price fully reflects all available information. The market is weak-form efficient if the current price of a security fully reflects all its information contained in its past prices, which means that studying the behaviours of historical prices cannot help in earning abnormal returns. The implication of weak-form efficiency is the random walk hypothesis (RWH), which indicates that successive price changes are random and serially independent. Stated differently, it is not possible to gain extra-normal profits from a market.

Weak form market efficiency implies that the markets are informationally efficient and there are no chances for extreme events such as a financial crisis. Any deviation will be temporary in such a set up. However, past evidences show that this is not the case. Financial markets are seen to be prone to extreme events, the latest being financial crisis of 2008. In such a situation, it will be of interest to study the behaviour of financial markets.

There are many methods to test weak form of market efficiency. Among them, variance ratio tests are considered powerful tools. Lo and MacKinlay (1988) were the first to construct the conventional variance ratio test. There were many improvements upon the Lo and MacKinlay (1988) test, and there were new tests constructed to address the potential shortcomings associated with the test. Chow and Denning (1993) modified Lo-MacKinlay's (1988) test to form a simple multiple variance ratio test. Choi (1999) put forward a data-driven automatic Variance Ratio test. Chen and Deo (2006) proposed a test to take care of the small sample distribution problem associated with Variance ratio (VR) statistic.

The efficient market hypothesis in the weak from is essentially deals with a linear dependency between current prices and past prices, while is perfectly possible that there could exist non-linear relationship between the variables under consideration. To analyse non-linear dependence in a time series, various tests are implemented. Some of the tools are BDS test (1987), Hinich Bispectrum test (1982) and neural network test by Teräsvirta (1993).

However, it is to be noted that none of the above mentioned test proposes a definite alternative hypothesis regarding the structure of the non-linearity. Non-linear systems could be stochastic or deterministic. Chaotic system is a type of non-linear deterministic system, with certain properties. Of the methods used in detection of chaos, Lyapunov exponents (LE) are powerful tools. It measures the rate of divergence between two near-by trajectories. If the value of LE for a system is positive, the system is said to be chaotic and vice versa.

Among financial markets of a country, forex markets hold great importance. A stable currency is a pre-requisite to economic growth and trade relations. Hence, it would be of great interest, especially from an economic policy view point, study the forex market behaviour. In an efficient market, there need not be any intervention by the monetary authority, as it will self correct. If the market is not efficient, the monetary authority could take necessary steps, as per the prevailing exchange rate management regime, if needed.

About BRICS

Jim O'Neill (2001) of Goldman Sachs coined the acronym 'BRIC'. The acronym has come into widespread use as a symbol of shift in global economic power towards the developing nations. Meetings between BRIC countries began in New York in September 2006, with a meeting of the BRIC foreign ministers. The BRIC countries met for their first official summit on 16 June 2009, in Yekaterinburg, Russia, with leaders of Brazil, Russia, India and China attending.

The core focus of the summit was the improvement of the current global economic situation and discussing how the four countries can better work together in the future, as well as a more general push to reformation in financial institutions. There were also discussions regarding how emerging markets, such as those members of BRIC, could be better involved in global affairs in the future.

South Africa sought membership during 2010, and was admitted as a member nation on December 24, 2010 after being formally invited by BRIC countries to join the group. The group was renamed BRICS to reflect the five-nation membership, with an "S" for South Africa appended to the acronym.

As BRICS aims at economic co-operation, and considering the fact that all of the member nations are developing countries, it will be of importance to see whether their respective economies follow any common traits. This study concentrates on the foreign exchange markets of BRICS countries, and seeks to analyze certain behaviours of the same. All of the forex markets follow managed float regimes with various degrees of intervention.

The reminder of this study is organized as follows. First, a brief review of the literature is presented. The analysis consists of three stages. In the first stage, the presence of weak form of market efficiency in the BRICS foreign exchange markets is analyzed using a family of Variance ratio tests. As the next stage, possible presence of non-linearity is analysed by employing BDS test (for raw series and residuals extracted from fitted GARCH models), Hinich Bispectrum test and Teräsvirta's Neural network test. The last stage of the analysis includes test to find a structure of non-linearity, if present. Estimation of Largest Lyapunov Exponent (LLE) was carried out for this purpose.

Literature Review

Tabak and Lim (2003) analyzed the random walk hypothesis on emerging markets exchange rates by employing Lo and McKinley (1989) variance ratio test on a daily and weekly frequency using a bootstrap technique, which is robust to heteroskedasticity. They examined some Asian and Latin American countries and Russia. Empirical evidence supports the random walk hypothesis on both daily and weekly frequencies for the recent period.

Das et al. (2007) has analyzed the presence of chaos in the foreign exchange markets of India and china by analyzing the bilateral daily exchange rates against US dollar. The methodology implemented was that of Largest Lyapunov Exponents. They estimated LLE's for the original exchange rate series as well as surrogate data series created from the original, and found the evidence of deterministic chaos in both the currencies.

Noman and Ahmed (2008) investigated the weak-form efficiency for foreign exchange markets in seven SAARC countries for the period from 1985 to 2005. They employed variance ratio test of Lo and MacKinlay (1988) and Chow-Denning joint variance ratio test (1993). Their study failed to reject the null hypothesis of random walk for all the seven currencies and the conclusion was that foreign exchange markets in South Asian region follow random walk process and, therefore, are weak-form efficient.

Asad (2009) tested the random walk and efficiency hypothesis for 12 Asia-Pacific foreign exchange markets using individual as well as panel unit root tests and variance ratio test of Lo and MacKinlay (1988) and the non-parametric-based variance ratio test of Wright (2000). The study used both daily and weekly spot exchange rate data from January 1998 to July 2007. While the daily data accepted the null hypothesis of weak form efficiency, it was rejected for the majority of the exchange rates while using weekly data.

Sasikumar (2011) analyzed the validity of weak form efficient market hypothesis in Indian foreign exchange market using 3 individual (Lo and McKinley, Wright, Choi) as well as 3 joint variance ratio(Chow-Denning, Chen-Deo and Wald) tests. All tests conclusively rejected the hypothesis of weak-form market efficiency.

Ibrahim et al. (2011) test for the weak form of market efficiency for the OECD countries efficiency using the Augmented Dickey-Fuller (ADF) and Phillip-Peron(PP)tests. The results indicate that the exchange rates studied follow random walks. The study used bilateral exchange rates to carry out the analysis.

Many of the above-mentioned studies were carried out using bilateral exchange rates and the analysis shows the evidence of weak-form market efficiency, except in the case of India and China. And regarding analysis of non-linearity and chaos for the exchange markets of said countries, there is a serious dearth of material, as per the best knowledge of the author, and it is the main motivation to carry out the present study. It could be said that bilateral exchange rates may not fully represent the dynamics of the foreign exchange market. Here, it is proposed to use Nominal Effective Exchange Rate (NEER) instead.

Data and Methodology

Monthly NEER data from April 1994-September 2014 is used for the analysis, taken from the Bank of International Settlement (BIS) website. To analyze the presence of weak-form market efficiency, 4 types of variance ratio tests (two individual, two joint) were employed. To test for non-linearity, BDS test, Hinich bispectrum test, and Teräsvirta neural network test were used. Largest Lyapunov exponent (LLE) were calculated for each series to test the presence of chaos. Detailed description of the tests is given in Appendix I.

Analysis

Tables 1 and 2 display the result of individual and joint variance ratio tests respectively. Tables 3, 4, 5 and 6 show the results of the tests for non-linearity. Table 7 displays the result of LLE estimation.

Table 1. *Result of individual variance ratio tests*

Test	Statistic	Holding Periods	Brazil	India	Russia	China	South Africa
Lo-Mac	Z_1	2	13. 3766 (0. 0000)	13. 55999 (0. 0000)	13. 34135 (0. 0000)	13. 65202 (0. 0000)	13. 73550 (0. 0000)
		5	23. 96705 (0. 0000)	22. 95078 (0. 0000)	22. 14875 (0. 0000)	23. 41138 (0. 0000)	23. 5504 (0. 0000)
		10	32. 19612 (0. 0000)	29. 67557 (0. 0000)	29. 32821 (0. 0000)	30. 25523 (0. 0000)	31. 1298 (0. 0000)
	Z_2	2	9. 33872 (0. 0000)	9. 276806 (0. 0000)	7. 468472 (0. 0000)	8. 073136 (0. 0000)	8. 59914 (0. 0000)
		5	16. 39696 (0. 0000)	16. 01941 (0. 0000)	12. 94002 (0. 0000)	14. 06835 (0. 0000)	15. 01571 (0. 0000)
		10	22. 6260 (0. 0000)	29. 67557 (0. 0000)	18. 18043 (0. 0000)	18. 70377 (0. 0000)	20. 52205 (0. 0000)
Choi	AV(K)		88. 95893 (0. 0000)	71. 3408 (0. 0000)	65. 30769 (0. 0000)	61. 57797 (0. 0000)	95. 0317 (0. 0000)

Note: for all the tables* indicates Significance at 1% level, P-values in the Parenthesis.

Table 2. *Result of joint variance ratio tests*

Test	Statistic	Holding Periods	Brazil	India	Russia	China	South Africa
Chen-Deo	QP_K	2, 5, 10	80. 24667*	76. 21783*	55. 59929*	58. 88748*	67. 35016*
Chow-Denning	CD_1	2, 5, 10	32. 1961*	29. 6755*	29. 3282*	30. 2552*	31. 1298*
	CD_2	2, 5, 10	22. 6260*	21. 4765*	18. 1804*	18. 7038*	20. 5220*

Table 3. *BDS test results (for the raw series)*

Currency/M&E	M=2, E=0. 5	M=4, E=1	M=8=E, 1. 5	M=10, E=2
Brazil	93. 0228 (0. 000)	76. 3654 (0. 000)	115. 4003 (0. 000)	70. 88 (0. 000)*
Russia	28. 3814 (0. 000)	39. 2551 (0. 000)	98. 3718 (0. 000)	132. 9022 (0. 000)
India	132. 1733 (0. 000)	85. 4309 (0. 000)	79. 4735 (0. 000)	46. 977 (0. 000)
China	60. 2393 (0. 000)	63. 1154 (0. 000)	59. 8894 (0. 000)	39. 566 (0. 000)
South Africa	95. 7388 (0. 000)	80. 2871 (0. 000)	77. 8248 (0. 000)	50. 456 (0. 000)

Table 4. *BDS test results (Residuals of the fitted GARCH Models)*

Fitted Models	M & E→	M=2, E=0. 5	M=4, E=1	M=8=E, 1. 5	M=10, E=2
Brazil (AR(1, 0) -EGARCH(1, 1))		5. 4751 (0. 000)	8. 0755 (0. 000)*	7. 9874 (0. 000)	5. 6085 (0. 000)
Russia (ARMA(1, 1)-EGARCH(1, 1))		13. 4818 (0. 000)	9. 7733 (0. 000)	7. 9728 (0. 000)	7. 9724 (0. 000)
India (AR(1, 0)-EGARCH(1, 1))		0. 3752 (0. 7075)	0. 1402 (0. 0883)	1. 9128 (0. 000)	2. 5535 (0. 010)
China (AR(0, 0) –APARCH(1, 1))		60. 0217 (0. 000)	62. 9749 (0. 000)	58. 83 (0. 000)	39. 04 (0. 000)
South Africa (AR(1, 0)-APARCH(1, 1))		3. 3537(0. 000)	1. 7831(0. 0743)	2. 2749(0. 0331)	0. 8709(0. 3814)

Table 5. *Teräsvirta Neural Network test result*

Currency\lag	2	3	4	5	6
Brazil	24. 09 (0. 000)	39. 267 (0. 001)	60. 15 (0. 000)	123. 03 (0. 001)	211. 75 (0. 000)
Russia	155. 42 (0. 000)	359. 24 (0. 000)	419. 37 (0. 000)	523. 75 (0. 000)	523. 75 (0. 000)
India	3. 06 (0. 86)	16. 44 (0. 442)	40. 28 (0. 009)	77. 71 (0. 007)	168. 70 (0. 003)
China	7. 74 (0. 378)	33. 67 (0. 006)	40. 20 (0. 09)	75. 96 (0. 010)	140. 92 (0. 000)
South Africa	12. 09 (0. 090)	17. 030 (0. 366)	37. 50 (0. 160)	61. 96 (0. 119)	110. 38 (0. 000)

Table 6. *Hinich Bispectrum test results*

Currency	H-statistics
Brazil	30. 53(0. 000)*
Russia	77. 54(0. 000)
India	53. 45(0. 000)
China	58. 74(0. 000)
South Africa	56. 49(0. 000)

Table 7. *Largest Lyapunov Exponent*

Currency	Largest Lyapunov exponent
Brazil	0. 2458
Russia	0. 1011
India	0. 0567
China	0. 0803
South Africa	0. 0605

The holding periods (k) for VR tests considered are (2, 5, 10) as advocated by Deo and Richardson (2003). After examining the Individual VR test results, we can see convergence among the various individual variance ratio tests. Here all the individual test statistics are significant at 1% level, and therefore the null hypotheses of weak-form market efficiency are rejected by all the tests.

After examining the joint VR test results, a similar conclusion is reached. Here too, a convergence among the test results is observed. All the test statistics are showing significance at 1% level. Hence, the null hypothesis is rejected with the joint tests also.

From the above results, it could be concluded that foreign exchange markets of the BRICS countries are not weak form efficient.

In the second stage, the presence of non-linearity was analyzed using various tests. The BDS test was applied in successive increasing embedding dimensions and increasing values of epsilon, for raw series, as well as residual extracted from fitted GARCH models. The GARCH models were fitted in order to see whether the non-linearity present could be explained by a GARCH process. While BDS test was applied to the raw dataset, the null hypothesis of i. i. d. was rejected for all values of m and epsilon for all series.

While the BDS test was applied to the extracted residuals from the GARCH models, the null of linear independence were rejected at all levels of m and epsilon for NEER series of Brazil, Russia and China. For India, the presence of non-linearity were confirmed for the values m = 8, epsilon = 1.5, and m = 10, epsilon = 2, and for South Africa, it was confirmed for m = 2, 4, 8 and corresponding epsilon = 0.5, 1, 1.5. This indicates that a GARCH process cannot satisfactorily explain the non-linearity present in these series, and further analysis is required.

The Teräsvirta Neural network test confirmed the presence of non-linearity for NEER of Brazil and Russia at all lags, while for the other three countries, the presence of non-linearity was detected in higher lags. The results from the Hinich bispectrum test also confirmed the presence of non-linearity in all of the five series. Thus it was confirmed non-linearity is present in all the five series under analysis after examining various test results.

To examine the nature of the non-linearity, LLE was calculated for each dataset. A positive value of Lyapunov exponent signifies chaotic behaviour. Here, the estimation returned positive values for all the five series. The value of Lyapunov exponent was highest for Brazil, indicating that the Brazilian forex market is most chaotic among the group. Indian forex market had the least value of Lyapunov exponent. Hence, it could be said that chaotic behaviour is present in the foreign exchange markets of "BRICS" countries in various degrees.

Conclusion

This study examined the possible presence of weak form of market efficiency, non-linearity and chaotic behaviour in the foreign exchange markets of Brazil, Russia, India, China and South Africa using a three-step-analysis. The first stage of analysis consisted of using variance ratio (VR) tests for checking weak form market efficiency. Presence of non-linearity was tested using BDS test, Hinich Bispectrum test and Teräsvirta Neural Network test. In the third stage, estimation of Largest Lyapunov exponents (LLE's) was carried out to detect the possible presence of chaos. The first stage rejected the Weak-form Efficient Market Hypothesis for all the five series. The next step was to analyze the possible presence of non-linearity. Here too, all tests confirmed the presence of non-linearity in the five series under study. The third step of the study, which was to analyze the structure of non-linearity, was carried out by estimating LLE for all five series. LLE's for all the 5 series returned positive values, and thus confirmed the presence of an underlying chaotic structure for all markets.

Here, we could assume that these markets may share some common characteristics, which resulted in them being not weak-form efficient, non-linear and chaotic. However, further study is required to identify the possible presence of such factors in order to reach at a strong conclusion.

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Appendix-I. Description of the tests

A. TESTS FOR WEK-FORM MARKET EFFICIENCY

1. Lo and MacKinlay (1988, 1989) Variance Ratio test

RWH for a time series X_t can be given by the following equation:

$$X_t = \mu + X_{t-1} + \varepsilon_t \tag{1}$$

μ is an arbitrary drift parameter and ε_t is the random disturbance term.

The underlying assumption is that the disturbance terms ε_t are independently and identically distributed normal variables with variance σ_o^2 . This is the assumption according to the traditional RWH.

Thus,

$$H : \varepsilon_t \text{ i. i. d. } N(0, \sigma_o^2) \tag{2}$$

According to the null hypothesis that the variance ratio should be unity for all levels of aggregation, it can be described as follows;

$$VR(q) = \frac{1}{q}\frac{\sigma_q^2(q)}{\sigma_1^2(q)} = 1 \tag{3}$$

The test statistic that is developed by Lo and MacKinlay for the variance ratio is as follows;

$$Z_1(q) = \sqrt{nq}\,\overline{M}_r(q)(2(2q-1)(q-1)/3q)^{-1/2} \overset{a}{\approx} N(0,1) \tag{4}$$

Where the variance ratio is,

$$\overline{M}_r(q) \equiv \frac{\overline{\sigma}_c^2(q)}{\overline{\sigma}_a^2} - 1 \tag{5}$$

And where the variance estimators are;

$$\overline{\sigma}_a^2 = \frac{1}{nq-1}\sum_{k-1}^{nq}(X_k - X_{k-1} - \hat{\mu})^2 \tag{6}$$

And,

$$\overline{\sigma}_c^2(q) = \frac{1}{m}\sum_{k=q}^{nq}(X_k - X_{k-q} - q\hat{\mu})^2 \tag{7}$$

Where,

$$m = q(nq - q + 1)(1 - \frac{q}{nq}) \tag{8}$$

The tests are based on different aggregation levels, signaled by q.

Next to the homoskedastic test statistic, Lo and MacKinlay (1989) also developed a test statistic that is robust to heteroskedasticity. They developed this test statistic with the knowledge that volatilities change over time, and that the error terms of financial time series are often not normally distributed.

Since $\overline{M}_r(q)$ still approaches zero, therefore we only have to calculate its asymptotic variance, which is defined as θ_q .

The variance ratio estimate as defined before, is asymptotically equivalent to a weighted sum of serial autocorrelation coefficient estimates, such that;

$$\overline{M}_r(q) = \sum_{j-1}^{q-1} \frac{2(q-j)}{q} \hat{\rho}(j) \tag{9}$$

Where $\hat{\rho}(j)$ is the estimator of the j^{th} autocorrelation factor.

Here, the asymptotic distribution of $\overline{M}_r(q)$ under the null hypothesisis defined as follows;

$$\sqrt{nq}\,\overline{M}_r(q) \overset{a}{\approx} N(0, V(q)), \tag{10}$$

Where $V(q)$ is the asymptotic variance of $\overline{M}_r(q)$ and can be calculated as

$$V(q) = \sum_{j=1}^{q-1} (2(q-j)/q)^2 \,\overline{\delta}(j), \tag{11}$$

Where

$$\overline{\delta}(j) = \frac{(nq) \sum_{k=j+1}^{nq} (X_k - X_{k-1} - \overline{X})^2 (X_{k-j} - X_{k-j-1} - \overline{X})^2}{\left[\sum_{k=1}^{nq} (X_k - X_{k-1} - \overline{X})^2 \right]^2} \tag{12}$$

And $\overline{\delta}(j)$ is the estimator for the weighted sum of the variances of $\hat{\rho}(j)$.

The standard normal Z-statistic under heteroscedasticity is computed as:

$$Z_2(q) = \sqrt{nq}\,\overline{M}_r(q) [V(q)]^{-1/2} \overset{a}{\approx} N(0,1). \tag{13}$$

2. Automatic variance ratio test under conditional heteroskedasticity of Choi (1999)

While implementing the VR tests, the choice of holding period *k* is important. However, this choice is usually rather arbitrary and *ad hoc*. To overcome this issue, Choi (1999) proposed a data-dependent procedure to determine the optimal value of k. Choi (1999) suggested a VR test based on frequency domain since Cochrane (1988) showed that the estimator of $V(k)$, which uses the usual consistent estimators of variance, is asymptotically equivalent to 2π times the normalized spectral density estimator at the zero frequency, which uses the Bartlett kernel.

However, Choi (1999) employed instead the quadratic spectral (QS) kernel because this kernel is optimal in estimating the spectral density at the zero frequency (Andrews, 1991). The VR estimator is defined as

$$VR(k) = 1 + 2\sum_{i=1}^{T-1} h(i/k)\,\hat{\rho}(i) \tag{14}$$

Where R(i) is the autocorrelation function, and $h(x)$ is the QS window defined as

$$h(x) = \frac{25}{12\pi^2 x^2}\left[\frac{\sin\left(\frac{6\pi x}{5}\right)}{6\pi x/5} - \cos\left(\frac{6\pi x}{5}\right)\right] \tag{15}$$

The standardized statistic is

$$VR_f = \frac{VR(k)-1}{(2)^{\frac{1}{2}}(T/k)^{-1/2}} \tag{16}$$

Under the null hypothesis the test statistic VR_f follows the standard normal distribution asymptotically. Note that it is assumed that $T \rightarrow \infty$, $k \rightarrow \infty$ and $T/k \rightarrow \infty$. Choi (1999) employed the Andrews (1991) methods to select the truncation point optimally and compute the VR test. Note that the small sample properties of this automatic VR test under heteroskedasticity are unknown and have not been investigated properly.

Joint variance ratio tests

1. Chow and Denning (1993) multiple variance ratio test

The test developed by Lo and MacKinlay (1988) uses the property of the RWH to test individual variance ratios for different values of the aggregation factor q. Chow and Denning (1993) recognized that the test lacks the ability to test whether all the variance ratios of the different observation intervals are equal to 1 simultaneously. This is a requirement of the RWH, and since Lo and MacKinlay (1988) overlooked this requirement, they used the standard normal tables to test the variance ratios on significance. Failing to control for the overall test size, leads to a large probability of a Type 1 error.

To circumvent this problem, Chow and Denning developed a test that controls for the joint test size, and also provides a multiple comparison of variance ratios. They used the Studentized Maximum Modulus (SMM) critical values to control for the overall test size and to create a confidence interval for the Variance Ratio estimates. They used the same test statistic of the Lo and MacKinlay (1988) Variance Ratio test. Only now they are simply compared to the SMM critical values, instead of the standard normal critical values to look for significance.

Since Chow and Denning (1993) consider multiple comparisons of the variance ratio estimates, and all variance ratio estimates should be above the SMM critical value, they use the following largest absolute value of the two test statistics as defined before in the Lo and MacKinlay (1988) procedure

$$Z_1^*(K) = \mathrm{Max}_{1\le i\le K}|Z_1(q_i)| \tag{17}$$

$$Z_1^*(K) = \mathrm{Max}_{1\le i\le K}|Z_1(q_i)| \tag{18}$$

$Z_1(q)$ and $Z_2(q)$ is calculated same as above

In which (q_i) are the different aggregation intervals for $\{q_i | i = 1,2,...,m\}$. The decision about whether to reject the null hypothesis or not can be based on the maximum absolute value of individual variance ratio test statistics.

2. Joint Variance Ratio Test of Chen and Deo(2006)

Chen and Deo (2006) suggested a simple power transformation of the VR statistic that, when k is not too large, provides a better approximation to the normal distribution in finite samples and is able to solve the well-known right-skewness problem. They showed that the transformed VR statistic leads to significant gains in power against mean reverting alternatives. Furthermore, the distribution of the transformed VR statistic is shown, both theoretically and through simulations, to be robust to conditional heteroscedasticity.

They defined the VR statistic based on the periodogram as

$$VR_p(k) = \frac{1}{1-{}^{k}/_{T}}\frac{4\pi}{T\hat{\sigma}^2}\sum_{j=1}^{(0.5(T-1))} W_k(\lambda_j)I_y(\lambda_j) \tag{19}$$

Where,

$$I_y(\lambda_j) = (2\pi T)^{-1}\left|\sum_{t=1}^{T}(Y_t - \hat{\mu})\exp(-i\lambda_j t)\right|^2 \tag{20}$$

$$\hat{\sigma}^2 = (T-1)^{-1}\sum_{t=1}^{T}(Y_t - \hat{\mu})^2 \tag{21}$$

And $\lambda_j = 2\pi j/T$; while $W_k(\lambda) = k^{-1}\{\sin(0.5k\lambda)/\sin(0.5\lambda)\}^2$ is a weighting function. Chen and Deo (2006) found that the power-transformed statistic $VR_p^{\beta_K}$ (k) gives a better approximation to a normal distribution than $VR_p(k)$, where

$$\beta_k = 1 - \frac{2}{3}\frac{\left(\sum_{j=1}^{(0.5(T-1))} W_k(\lambda_j)\right)\left(\sum_{j=1}^{(0.5(T-1))} W_k^3(\lambda_j)\right)}{\left(\sum_{j=1}^{(0.5(T-1))} W_k^2(\lambda_j)\right)^2} \tag{22}$$

Let $(k_1, \ldots, k_l)$ be a vector of holding periods satisfying the conditions given in Theorem 5 of Chen and Deo (2006). Conditions (A1) to (A6) in Chen and Deo (2006) allow the innovations ε_t to be a martingale difference sequence with conditional heteroskedasticity. They are explained below.

A1) $\{\varepsilon_t\}$ is ergodic and $E(\varepsilon_t/\vartheta_{t-1}) = 0$ for all t, where ϑ_t is a sigma field, ε_t is ϑ_t measurable And $\vartheta_{t-1} \subset \vartheta_t$ for all t.

(A2) $E(\varepsilon_t^2) = \sigma^2 < \infty$

(A3) For any integer q, $2\leq q \leq 8$ and for q non negative integers s_i, $E(\prod_{i=1}^{q} \varepsilon_{t_i}^{s_i})=0$ when at least one s_i is exactly one and $\sum_{i=1}^{q} s_i \leq 8$

(A4) For any integer r, $2\leq r \leq 4$ and for r non negative integers s_i, $E(\prod_{i=1}^{q} \varepsilon_{t_i}^{s_i}/\vartheta_t)=0$ when at least one s_i is exactly one and $\sum_{i=1}^{r} s_i \leq 4$

(A5)$\lim_{n-\to\infty} Var(E(\varepsilon_{t+n}^2 \varepsilon_{t+n+j}^2 | \vartheta_t)) = 0$ uniformly in j for every j>0

(A6) $\lim_{n-\to\infty} E(\varepsilon_t^2 \varepsilon_{t-n}^2) = \sigma^4$

Under the assumption that given time series Y_t follows a conditionally heteroskedastic martingale difference sequence Chen and Deo Showed that

$$V_{p,\beta} \equiv (VR_p^{\beta_1}(k_1), \ldots\ldots, VR_p^{\beta_l}(k_l))' \tag{23}$$

approximately follows $N(\mu_\beta, \Sigma_\beta)$. The details of μ_β and Σ_β are given in Chen and Deo (2006). Based on this, Chen and Deo (2006) proposed a joint test statistic of the form

$$QP = (V_{p,\beta} - \mu_\beta)' \Sigma_\beta^{-1} (V_{p,\beta} - \mu_\beta) \tag{24}$$

It approximately follows a chi-squared distribution with l degrees of freedom under H_0: $V(k_1) = \ldots = V(k_l) = 1$ against H_1: $V(k_i) \neq 1$ for some i.

B. TESTS FOR NON-LINEARITY

1. BDS (1987)Test

W.A. Brock, W. Dechert and J. Scheinkman proposed BDS test in 1987 (Brock et al., 1987). BDS test is a powerful method for detecting serial dependence in time series. It tests the null hypothesis of independent and identically distributed (I. I. D.) against an unspecified alternative.

The BDS test is based the correlation integral concept. Consider a time series $\{x_t$: $t = 1, 2, \ldots, N\}$, which is a random sample of independent and identically distributed (i. i. d.) observations. The correlation integral $C_m(\varepsilon)$ measures the probability that any two of the points $\{X_i\}$ meet within distance ε from each other in m dimensional phase space, and must equal to the product of the individual probabilities, provided that pairs of points are independent:

$$C_m(\varepsilon) = \prod_{i,j(i\neq j)} p(||X_i - X_j|| < \varepsilon) \; for \; N \to \infty \tag{25}$$

if all observations are also identically distributed, then

$$C_m(\varepsilon)=(C_1(\varepsilon))^m \text{ for } N\to\infty. \tag{26}$$

The statistic: $B(m, \varepsilon, N)=N^{0.5}(C_m(\varepsilon)\text{-}C_1(\varepsilon))$ (27)

would converge to a normal distribution with zero mean and a variance $V(m, \varepsilon, N)$ which could be consistently estimated from the sample data. The BDS statistic is defined as

$$W(m, \varepsilon, N) = \frac{B(m,\varepsilon,N)}{(V(m,\varepsilon,N))^{1/2}} \quad for \; N \to \infty \tag{28}$$

The BDS statistic, W, will follow a standard normal distribution. The null hypothesis of BDS test is the testing series is of i. i. d. observations. If the W estimator is larger than the level of significance, we can reject the null hypothesis, that is, the nonlinearity exists in the testing series.

2. Hinich (1982)Bispectrum test:

The Hinich (1982) bispectrum test is a frequency domain test. It estimates bispectrum of stationary time series and provides a direct test for non-linearity in the given series. The flat skewness indicates that the return generating process is linear. In other words, the test checks for third order non-linear dependence.

For frequencies ω_1 and ω_2 in the principal domain given by

$$\Omega = \{(\omega_1, \omega_2): 0 < \omega_1 < 0.5, \omega_2 < \omega_1, 2\omega_1 + \omega_2 < 1\}, \qquad (29)$$

the bispectrum $B_{xxx}(\omega_1, \omega_2)$ is defined by

$$B_{xxx}(\omega_1, \omega_2) = \sum_{r,s=-\infty}^{\infty}(C_{xxx}(r, s)\text{exp}\,(-i2\pi(\,\omega_1 r + \omega_2 s)) \qquad (30)$$

The bispectrum is the double Fourier transformation of the third-order moments function and is the third-order polyspectrum. The regular power spectrum is the second-order polyspectrum and is a function of only one frequency.

The skewness function $\Gamma(\omega_1, \omega_2)$ is defined in terms of the bispectrum as follows:

$$\Gamma^2(\omega_1, \omega_2) = \frac{|B_{xxx}(\omega_1,\omega_2)|^2}{S_{xx}(\omega_1)S_{xx}(\omega_2 S_{xx}(\omega_1+\omega_2)} \qquad (31)$$

where $S_{xx}(\omega)$ is the (ordinary power) spectrum of $x(t)$ at frequency ω. Since the bispectrum is complex valued, the absolute value in Equation 31 shows modulus. Brillinger (1965) proved that the skewness function $\Gamma(\omega_1, \omega_2)$ is constant over all frequencies $(\omega_1, \omega_2)\epsilon\Omega$ if $\{x(t)\}$ is linear; while $\Gamma(\omega_1, \omega_2)$ is flat at zero over all frequencies if $\{x(t)\}$ is Gaussian. Linearity and Gaussianity can be tested using a sample estimator of the skewness function.

3. Teräsvirta (1993) neural network test

In Teräsvirta (1993) neural network test, the time series is fitted with a single hidden-layer feed-forward neural network, which is used to determine whether any nonlinear structure remains in the residuals of an autoregressive (AR) process fitted to the same time series. The null hypothesis for the test is 'linearity in the mean' relative to an information set. A process that is linear in the mean has a conditional mean function that is a linear function of the elements of the information set, which usually contains lagged observations on the process.

The intuition behind Teräsvirta test can be summed as follows: under the null hypothesis of linearity in the mean, the residuals obtained by applying a linear filter to the process should not be correlated with any measurable function of the history of the process. Teräsvirta test uses a fitted neural net to produce the measurable function of the process's history and an AR process as the linear filter. It then tests the hypothesis that the fitted function does not correlate with the residuals of the AR process. The resulting test statistic has an asymptotic χ^2 distribution under the null of linearity in the mean.

C. Test for CHAOS

1. Largest Lyapunov Exponent (LLE)

Analysis of the chaotic behavior depends on the concept of sensitive dependence to initial conditions (SDIC) and the opinion that chaos will exist if nearby trajectories diverge exponentially. One of the implications the existence of SDIC is the systematic loss of predictability of the system over time. The notion of Lyapunov spectrum is often used to quantify and detect this phenomenon.

Lyapunov exponents are calculated as follows:

$$\lambda = \lim_{n\to\infty} \ln \left(||Df^n(x)\vec{v}||\right)/n \qquad (32)$$

where D signifies the derivative, $//\,//$ is the Euclidian norm, f^n is the nth iteration of dynamical system f with initial conditions in point x and $\vec{v}$ is a direction vector. If the largest real part of these exponents is positive then the system exhibits sensitivity to initial conditions. In such a case, a larger magnitude means faster decay in the predictability.

This method requires knowledge of the analytical structure of the underlying dynamics. In cases where the true dynamics are not known, the alternative is to devise methods for extracting information about the rates of divergence between nearby orbits from a sequence of observed data. An algorithm suggested by Wolf et al. (1985) has been used for this purpose. Here, a slightly modified version of Wolf's algorithm suggested by Kantz (1994) is used for the analytical purpose.

The procedure could be explained by defining a line S, as a function of the number of time steps, number of observations, the embedding dimension and radius of the ball B (which is an indicator for the size of the neighborhood):

$$S(\Delta n, N, m, \varepsilon) = \frac{1}{N-m+1}\sum_{i_0=1}^{N-m+1} ln\left(\frac{1}{|B(X_{i0})|}\sum_{X_j \epsilon B(X_{i0})} ||x_{(i_0+\Delta n,1)} - x_{(j+\Delta n,1)}||\right) \qquad (33)$$

Where, $|B(\cdot)|$ is the total number of neighbors in the neighborhood B (a ball with diameter ε) of the reference vector X_{i0}. $x_{(i0,\,1)}$ is the most recent, element in the reference vector, X_{i0} and $x_{(i0+\Delta n,\,1)}$ is the first observation outside the time span covered by the reference vector.

The basic idea is to trace the distance in between a reference point X_0 and its neighbor, X_j, after n time steps. Set $d_j(X_0, X_j, n)$ to be this distance in the reconstructed phase space and let $\varepsilon(X_0, X_j)$ denote the initial distance between X_0 and X_j. In this case, $d_j(X_0, X_j, n)$ should grow exponentially by the largest Lyapunov exponent $\lambda_{\max}(X_0)$, or as it might be expressed in logarithm scale

$$ln\, d_j(X_0, X_j, n) \approx \lambda_{max}(X_0)n + ln\varepsilon(X_0, X_j) \qquad (34)$$

It is proposed that, if this linear pattern is persistent for a number of time steps n, the estimated slope is an estimate for the largest Lyapunov exponent. Kantz procedure keeps track of all neighbors within a neighborhood ball $B(X_0)$. By taking an average of all neighbors within the neighborhood, this method seems more robust against noisy elements.