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Trade in a two sector endogenous growth model with two accumulating factors

Birgit KIRSCHBAUM-BEHL
University of Greifswald, Germany
birgit.kirschbaum@uni-greifswald.de

Abstract. *This paper provides an endogenous growth model including technology diffusion induced by openness. A two sector model composed of a consumption good sector and an education sector shows that the openness leads to a compensation of the lack considering the human capital stock. It is a comparison between a relatively less developed country, which does not use physical capital in the education sector, and a relatively developed country using both, human and physical capital in the education sector. By importing a relatively human capital intensive good, the relatively less developed country benefits from technology diffusion. This paper shows, that introducing trade implies a higher growth rate than in a closed economy.*

Keywords: endogenous growth, accumulating factors, trade.

JEL Classification: D51, D91, F10, O11.

1. Introduction

Inequality is a useful incentive to economic activities. Not only the big differences in the world can be an engine for free trade but also a slight imbalance may be responsible for large consequences in similar regions. The goal of this paper is to analyze the effects of free trade between industrialized developed countries on their economic growth.

Motivated by the role of the EU there is the question if trade affects a relatively homogenous area like Europe. For many people Europe embodies a coalescent example which stands for freedom, wealth and culture. As similar as the countries may appear, actually they are quite different. For a successfully constant economic growth Europe countries have to grow together strongly, despite all versatilities. Does free trade, induced by these inequalities, support the globalization? Is there an advantage of trade considering the economic growth rate of the Member States?

To answer these questions this work looks at a model of the neoclassical growth theory, which is expanded to an additional trade component. As expected trade could have a similar effect like an exogenous shock with a subsequently convergence to the world markets steady state. By having a closer look on the education sector of a country it is questionable if varieties in this sector cause trade and if so does free trade affect economic growth?

The latest events of the last years have shown that there is still a transition between success and setbacks. In 2012 Spain reduced the investments in the education sector, as part of a cost-cutting program. This implicates less money for teachers by working more hours a day and even the funds of research institutes and universities has been shortened. Another engine of economic growth is the human capital accumulation as known from Lucas (1988).

The present work is based on the Lucas-Uzawa Model of endogenous growth, which shows the importance of human capital accumulation for economic growth. Romer (1990) and Barro (1991) contributed the empirical confirmation of the important and distinct role of human capital.

Galor and Mountford (2006, 2008) showed in theory and evidence, that trade causes the Great Divergence in income per capita for industrialized and non-industrialized economies. They argue, that the former comparative advantages shaped the trade pattern and thus the development divergences. A comparative advantage in the production of labor-intensive good leads to a specialization in that sector in particular. Therefore, there is a higher incentive to invest in the size of population. Differently from the present industrialized countries they had a comparative advantage for capital intensive goods, resulting a specialization in capital intensive goods. Assuming, that labor intensive goods are produced with low skill labor and capital-intensive goods with high skill labor, the industrialized economies concentrated on the investments in education to generate high skill labor.

The different investment approaches influence the income per capita in contradictory ways. More developed industrialized countries invest in human capital accumulation and thus the impact of trade is a higher income per capita. Less developed countries focus on the size of population and in turn the income per capita drops.

But even this theory does not consider physical capital as a factor of production for human capital accumulation. The following models include physical capital as an input of the human capital accumulation like in Bond et al. (2003), Mino (1996), Mulligan and Sala-i-Martin (1993), Stokey and Rebelo (1995). They showed that expenditures in goods and physical capital account for the human capital technology. To open an economy to the rest of the world implies changes of supply and demand at the factors and goods markets. The above-mentioned expenditures can be compensated or interpreted as the additional welfare caused by free trade

By looking only on economics with a similar development stages it is interesting that they use for the education sector as well as for the manufacturing both factors of production. This growth model is an endogenous growth model with two accumulating factors, physical and human capital. But the trade literature has concentrated almost exclusively on physical capital as the only accumulation factor. Bond et al. (2003) created one of the first models with trade in an endogenous growth model with two accumulating factors.

The model of Lucas (1988) neglected the role of physical capital in the human capital sector. The models of the paper "Equilibrium dynamics in two-sector models of endogenous growth" written by Ladrón-de-Guevara, Ortigueira and Santos (1997) close this gap in the literature. It is another essential example for this paper. On the one hand an equilibrium which includes physical capital as an input to the education sector is analyzed and on the other hand they show the positive influence of leisure in the utility function on the equilibrium.

The present paper looks only at the first model and questions if there is an effect of trade on the balanced growth path. Is the state of development crucial for the impact of trade between similar regions?

But the following work shows, that trade induce a convergence of income per capita to the development stages. A less industrialized country exports physical capital and is able to invest all available human capital in education. Beside this trade generate an additional human capital increase by importing human capital-intensive goods. Unlike a relatively more developed country, there is a negative effect of trade on human capital accumulation. The specialization in the consumption sector reduces the investment in human capital and there is also a loss of human capital by exporting human capital indirectly and thus reducing the development gap. Here the decisive reason for trade differs. It is based on different endowment ratios of the countries, with the same technological standard.

To sum up: this model combines the both most influential factors for economic growth, human capital accumulation and trade.

The paper is organized as follows. It begins with a description of the autarky model in section 2 by taking a closer look to the production and consumption of the considered economy. Section 3 deals with the introduction of trade and differ between the two types of countries. First a less developed country is looked at and a more developed country is examined afterward. Section 4 concludes.

2. Intuition

The example of the textile industry should simplify the construction and mechanism of the model, more precisely the production process of a handbag. When we look at the small economy there is a consumption sector and an education sector. The consumption sector produces a consumption good which benefits the households by a given utility. Here the handbag should satisfy the demand of the consumers. In particular the special need of the consumers is to carry something from one place to another. This commodity good can be produced in different ways depending on the ratio of production factors. In general, human capital and physical capital are needed for the production process. On the one hand it can be produced with relatively more human capital or on the other hand relatively physical capital intensive.

On the first case a bag is considered which is handmade and accurate in every detail, such as the Hermes Birkin Bag, which is a classical design item of France. The production process is very complex and needs high skilled bag makers. Thus, relatively more human capital is needed for the manufacture

Having a closer look at the relatively physical capital-intensive production process, an imitation of the mentioned bag can be produced with relatively more machines than specific tailors knowledge. A country like Bangladesh has workers which are skilled in sewing and are also used to sewing machines. Even the cuttings of the pattern are mainly made by machines. In contrast to France the pieces are cut by tailors their selves.

The same example is used for the education sector. The produced good is the education of the workers, like skills and abilities. There are also two different ways how to accumulate human capital. The workers can learn how to sew and produce the bag only with the help of a teacher. Here human capital is the exclusively production factor in the education sector. The workers learn while listening to the teacher and get an imagination of how it would work.

If both factors are used in the education sector, then the teacher training is supported with a sewing machine, for example. Now then the workers are able to observe the manufacture and know more precisely the processing steps. It is also conceivable that technical equipment supports the learning process. A TV or tablet PC could be used to watch some tutorial videos. Even in this case human capital and physical capital is used in the production process.(1)

This paper looks at a small country which trades with the world market at the time of different development stages. A country is less developed, as soon as the education sector is less developed compared to the rest of the world. For more developed countries the contrary is the case. The impact of the various development stages appears in the production processes in each sector.

A less developed country has relatively more physical capital than human capital. That is why the production of the consumption good is physical capital-intensive. A closer look to the education sector which is relatively less developed shows, that human capital is the only production factor.

Unlike the situation which dominates a relatively more developed country. Both factors are used for both sectors. But the countries' consumption sector uses relatively more human capital than physical capital compared to the world market. Bangladesh produces the handbag in a completely different way than France does. It starts with the education of the workers and end up with the manufacturing itself.

So far that is the situation without any trade. These differences lead to trade with the consumptions good. The idea is that a country could benefit from the import of the consumption good by analyzing this product. Especially the human capital accumulation is stimulated by trade. Here the handbag could be measured out exactly, the nature and position of the seams could be determined and also the nature and quality of the material could perfect the imitation. The processed human capital is a new source for the education sector.

3. Autarky

This paper is focused on a general model economy with endogenous growth. Starting with an autarkic economy two kinds of agents are assumed in one observed country: representative identical firms and representative identical households. The two sector model consists of the education and the production sector of a consumption good. There are no transport costs or market barriers in the considered country.

3.1. Production

Generally, there are two factors of production, labor $\boldsymbol{L}$ and capital $\boldsymbol{K}$. They can be used for the consumption and education sector and can move freely between these sectors thus they face to the same factor prices. The factor price of capital is the interest $\boldsymbol{r}$ and the rate of return for the supplied labour is the wage $\boldsymbol{w}$. There are some more details about the production factors.

$$L = N(t)u(t)h(t)$$

Labor depends on the average amount of human capital $\boldsymbol{h(t)}$, $\boldsymbol{u(t)}$ is the amount of human capital which is used for the production of consumption goods and the population size is $\boldsymbol{N(t)}$.

$$K = v(t)k(t)$$

Capital consists of the average amount of physical capital $\boldsymbol{k(t)}$ and the portion which is used in the consumption sector $\boldsymbol{v(t)}$.

The concave production function for the consumption sector is $\boldsymbol{F[K,L] = A(v(t)k(t))^{\alpha}(u(t)h(t))^{1-\alpha}}$ which is increasing and linear homogeneous in both factors of production. This is the same for the production function of the education sector: $\boldsymbol{G[K,L] = B((1-v(t))k(t))^{\eta}((1-u(t))h(t))^{1-\eta}}$. Firms borrow two kinds of capital from the households, physical and human capital. The cost of producing the consumption good can be expressed as a function of $\boldsymbol{w}$ and $\boldsymbol{r}$, $\boldsymbol{P = p(w,r)}$. The profit maximization problem for the consumption sector at the factor and at the goods market is:

$$\boldsymbol{max\Pi = F[v(t)k(t), u(t)h(t)] - rv(t)k(t) - wu(t)h(t)} \quad (1)$$

$$\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial k(t)}} = \boldsymbol{rv(t)} \quad (2)$$

$$\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial h(t)}} = \boldsymbol{wu(t)} \quad (3)$$

The factors of production are substitutable and the same good can be produced physical capital intensive of a relatively less developed country or human capital intensive in a relatively more developed country.

3.2. Households

The households $\boldsymbol{N(t) = N_0 e^{nt}}$ grow with an exogenously rate $\boldsymbol{n}$. These consumers are factor owner and supply both physical capital and human capital to the factor markets as mentioned before. They choose between the accumulation of the produced good or to sell them for consumption. Therefore, the income is used to buy consumption goods and to invest in human capital accumulation. All consumers have the same preferences and discount their future utility with the rate $\boldsymbol{\rho}$.

$$\boldsymbol{V = \int_0^\infty e^{-\rho t}\psi\big(c(t)\big)\,dt} \quad (4)$$

and

$$\psi\big(c(t)\big) = \frac{c(t)^{1-\sigma}}{1-\sigma} \quad (5)$$

The consumption affects the current utility. For simplicity the effect of education on the households' utility will be ignored.

The variables of choice for the optimization problem is the amount of consumption $\boldsymbol{c(t)}$, the amount of physical capital allocated to production $\boldsymbol{v(t)}$ and education $\boldsymbol{(1 - v(t))}$ as well as the amount of human capital allocated to the production sector $\boldsymbol{u(t)}$ and the education sector $\boldsymbol{(1 - u(t))}$ and the maximized stream of discounted utility.

$$\max_{c(t),u(t),v(t)} V[k_0, h_0] = \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \quad (6)$$

subject to

$$\partial k/\partial t = \dot{k}(t) = A(v(t)k(t))^\alpha (u(t)h(t))^{1-\alpha} - c(t) \quad (7)$$

$$\partial h/\partial t = \dot{h}(t) = B((1 - v(t))k(t))^\eta ((1 - u(t))h(t))^{1-\eta} \quad (8)$$

$$c(t) \geq 0, \quad k(t) \geq 0, \quad h(t) \geq 0,$$

$$0 \leq u(t) \leq 1, \quad 0 \leq v(t) \leq 1,$$

$$0 \leq \alpha \leq 1, \quad 0 \leq \eta \leq 1, \quad A \geq 0, \quad B \geq 0$$

$$\rho > 0, \quad k_0, h_0 \quad exogenous$$

The Hamiltonian for the representative household is

$$\begin{aligned} \mathbb{H} = \quad & e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma} \\ & +\gamma_1(A(vk)^{\alpha}(uh)^{1-\alpha}-c) \\ & +\gamma_2 B[(1-v)k]^{\eta}[(1-u)h]^{1-\eta} \end{aligned} \tag{9}$$

$$\boldsymbol{\gamma_1 > 0}, \quad \boldsymbol{\gamma_2 > 0}$$

where $\boldsymbol{\gamma_1}$ and $\boldsymbol{\gamma_2}$ are the shadow prices of physical and human capital in utility units.

The first order conditions for the optimal solution are

$$\frac{\partial \mathbb{H}}{\partial c} \overset{!}{=} 0 \tag{10}$$

$$\frac{\partial \mathbb{H}}{\partial v} \overset{!}{=} 0 \tag{11}$$

$$\frac{\partial \mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma_1} \tag{12}$$

$$\frac{\partial \mathbb{H}}{\partial u} \overset{!}{=} 0 \tag{13}$$

$$\frac{\partial \mathbb{H}}{\partial h} \overset{!}{=} -\dot{\gamma_2} \tag{14}$$

$$\gamma_1 = e^{-\rho t}c^{-\sigma} \tag{15}$$

$$\gamma_1 A\alpha v^{\alpha-1}k^{\alpha}(uh)^{1-\alpha} = \gamma_2 B\eta(1-v)^{\eta-1}k^{\eta}[(1-u)h]^{1-\eta} \tag{16}$$

$$\gamma_1 A v^{\alpha}k^{\alpha-1}\alpha(uh)^{1-\alpha} + \gamma_2 B(1-v)^{\eta}k^{\eta-1}\eta[(1-u)h]^{1-\eta} = -\dot{\gamma}_1 \tag{17}$$

$$\gamma_1 A(1-\alpha)(vk)^{\alpha}h^{1-\alpha}u^{-\alpha} = \gamma_2 B(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta}h^{1-\eta} \tag{18}$$

$$\gamma_1 A(1-\alpha)(vk)^{\alpha}u^{1-\alpha}h^{-\alpha} + \gamma_2 B(1-\eta)[(1-v)k]^{\eta}(1-u)^{1-\eta}h^{-\eta} = -\dot{\gamma}_2 \tag{19}$$

$$\lim_{t\to\infty} e^{-\rho t}\gamma_1 k = 0 \tag{20}$$

$$\lim_{t\to\infty} e^{-\rho t}\gamma_2 h = 0 \tag{21}$$

This system of equations describes the optimal solution for the general equilibrium. Integrating the condition for Pareto efficiency for consumption demand $\boldsymbol{c(t)}$ and the inter temporal arbitrage conditions for physical and human capital as well as the transversality conditions at infinity. It arises a Keynes-Ramsey Rule for the autarky situation.

$$\hat{c} = \frac{1}{\sigma}\left(A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1} - \rho\right) \tag{22}$$

Model dynamics

Equation (22) shows, if the relation $\boldsymbol{k/h}$ and the amount of capitals $\boldsymbol{u}$ and $\boldsymbol{v}$ are constant, then the consumption rate should be constant as well. The growth of physical capital, the ratio $\boldsymbol{k/h}$ and capital amounts $\boldsymbol{v}$ and $\boldsymbol{u}$ are relevant in the same way. In addition, there is a consumption-physical capital ratio $\boldsymbol{c/k}$ or $\boldsymbol{\chi}$.(2) Physical capital $\boldsymbol{k}$ grows with a constant rate when $\boldsymbol{v}$ and $\boldsymbol{u}$ are constant too.

$$\hat{k} = Av^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} - \frac{c}{k} \qquad \widehat{=} \qquad \hat{k} = Av^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} - \chi \tag{23}$$

The situation is very similar for human capital. Again, $\boldsymbol{k/h}$, $\boldsymbol{v}$ and $\boldsymbol{u}$ are needed to be constant, otherwise human capital will not grow with a constant rate.

$$\hat{h} = B\left[(1-v)\frac{k}{h}\right]^{\eta}(1-u)^{1-\eta} \tag{24}$$

Shortened by $\boldsymbol{x_1} = \frac{\boldsymbol{vk}}{\boldsymbol{uh}}$ and $\boldsymbol{x_2} = \frac{(\boldsymbol{1-v})\boldsymbol{k}}{(\boldsymbol{1-u})\boldsymbol{h}}$ it follows:

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi \tag{25}$$

$$\hat{h} = Bx_2^{\eta}(1-u) \tag{26}$$

A steady state requires $\boldsymbol{\hat{c}} = \boldsymbol{\hat{k}} = \boldsymbol{\hat{h}}$ hence the growth rates are equal, if $\boldsymbol{k/h}$, $\boldsymbol{c/k}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are constant. An economy is only in an equilibrium if the ratio of marginal products of both sectors correspond with each other. This is deducible from (11) and (12).

$$\frac{1-\alpha}{\alpha}x_1 = \frac{1-\eta}{\eta}x_2 \tag{27}$$

An additional unit of physical or human capital leads in both sectors to the same marginal productivity. This holds true, if the ratios of the shadow prices of both sectors are equal to each other, thus the lifetime marginal products of the lifetime utility is the same. In the long run the relations $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ behave in the same way. The following equation (29) leads to the computation of both growth rates $\boldsymbol{\hat{x}_1}$ and $\boldsymbol{\hat{x}_2}$, which are equal in the equilibrium.

$$\hat{x}_1 = \hat{x}_2 = 0 \tag{28}$$

To solve the system of equations a further condition is necessary to affiliate (12) and (14). This condition describes, first that the rate of the marginal product of lifetime utility of human capital is equal with the marginal product of human capital. Second, the growth rate of the shadow price of physical capital is equal to the marginal product of physical capital in the consumption sector.

$$B(1-\eta)x_2^{\eta} = -\rho - \sigma\hat{c} - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{29}$$

The utility of an additional unit human capital in the education sector is the same utility of an additional unit physical capital, because $\boldsymbol{\rho - \sigma\hat{c} = \widehat{\gamma_1} = -A\alpha x_1^{\alpha-1}}$ according to (17) is true. Finally, the conditions for the equilibrium can be written as:

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi \tag{30}$$

$$\hat{h} = Bx_2^{\eta}(1-u) \tag{31}$$

$$x_1(1-\alpha)/\alpha = x_2(1-\eta)/\eta \tag{32}$$

$$\hat{c} = \frac{1}{\sigma}(A\alpha x_1^{\alpha-1} - \rho) \tag{33}$$

$$B(1-\eta)x_2^{\eta} = -\rho - \sigma\hat{c} - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{34}$$

For the solution of these equations, the optimal values of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are neccesary. Therefore, the growth rate of consumption is transformed to:

$$\hat{c}^* = \frac{1}{\sigma}\left(\left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}B^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}} - \rho\right) \tag{35}$$

The marginal product $\boldsymbol{\left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}B^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}}}$ depends on the technology parameters $\boldsymbol{A}$ and $\boldsymbol{B}$ and will be shortened with $\boldsymbol{M}$. By any technological progress economic growth increases.

The optimal allocation of human capital $\boldsymbol{u}^*$ follows from $\boldsymbol{\hat{c} = \hat{h}}$.

$$u^* = \frac{\sigma M - (1-\eta)(M-\rho)}{\sigma M} \tag{36}$$

The decreasing marginal product of capital can be compensated by a higher use of human capital in the consumption sector. If an additional sewing machine is less productive than the one used before, the high-skilled labor will be used in the education sector instead of the consumption sector. With a higher discount rate $\boldsymbol{\rho}$ the investment of human capital decreases in the education sector. A higher utility in the future through better education is less relevant than current utility. The amount $\boldsymbol{u}^*$ is a reference value and should explain the openness of a country in the following part of the paper. The decreasing marginal product of capital can be compensated by a higher use of human capital in the consumption sector. If an additional sewing machine is less productive than the one used before, the high-skilled labor will be used in the education sector instead of the consumption sector. With a higher discount rate $\boldsymbol{\rho}$ the investment of human capital decreases in the education sector. A higher utility in the future through better education is less relevant than current utility. The amount $\boldsymbol{u}^*$ is a reference value and should explain the openness of a country in the following part of the paper. The optimal allocation of a closed economy in the equilibrium for physical capital is:

$$v^* = \frac{\alpha(1-\eta)\left(\frac{M}{B(1-\eta)}\right)^{\frac{1}{\eta}}(M\sigma-(1-\eta)(M-\rho))}{(1-\alpha)\eta M\sigma\left(\frac{\alpha(1-\eta)\left(\frac{M}{B(1-\eta)}\right)^{\frac{1}{\eta}}(M\sigma-(1-\eta)(M-\rho))}{(1-\alpha)\eta M\sigma}+\left(\frac{M}{B(1-\eta)}\right)^{\frac{1}{\eta}}\left(1-\frac{M\sigma-(1-\eta)(M-\rho)}{M\sigma}\right)\right)} \tag{37}$$

The surplus of physical capital will be used in the education sector and raises the productivity of this sector. The relation consumption-physical capital $\boldsymbol{\chi}$ can be determined by $\hat{\boldsymbol{c}} = \hat{\boldsymbol{k}}$.

$$\chi^* = \frac{1}{\sigma}\left(\frac{A\alpha\sigma[-\eta\rho+M(\eta+\sigma-1)+\rho]\left(\frac{\alpha(\eta-1)\left(\frac{M}{B(1-\eta)}\right)^{\frac{1}{\eta}}}{(\alpha-1)\eta}\right)^{\alpha-1}}{\rho(\alpha-\eta)+M(\alpha(\sigma-1)+\eta)}-M+\rho\right) \tag{38}$$

4. Trade

A main point of this work is the indirect transfer of human capital by international trade. Depending on the resulting trade structure the imports and exports support the education sector alongside. This effect is the spillover-effect of knowledge. By importing human capital-intensive produced goods, there is a transfer of knowledge, which could lead to a higher imitation rate. In the following model the parameter of diffusion or openness is $\overline{\boldsymbol{B}}$. By importing consumption goods $\boldsymbol{c_{im}}$ the country could absorb knowledge by analysing the imports. This rate of absorption is expressed by $\boldsymbol{\phi}$. Even if there is a unique level effect for the education sector, the possibility of developing imitations and reduce the imports increases as well.

The trade model consists of two different regions, the world market "WM" and a home country "*". In the following part two different development stages are looked at. The considered country is economical small. This means that the decisions of the country do not affect the world market. The agents of both regions have the same preferences and work with the same technologies. There are still no transport costs and no market barriers. The wide variation considers the endowment with factors of production. More precisely the world market has an absolute advantage in both production factors, physical and human capital, but the physical capital - human capital ratio $\boldsymbol{k/h}$ differs in both regions.

$$(k(t)/h(t))^{WM} \neq ((k(t)/h(t))^* \tag{39}$$

Two cases are distinguished: The world market trades with either a relatively less developed country or with a relatively more developed country. Physical capital is tradeable and the only mobile factor of production. The other commodity is the consumption good. In the open economy as well as in the world market are identical households $\boldsymbol{N^i(t) = N_0^i e^{nt}}$. The equations of the two factors of production in each country $\boldsymbol{i}$ are: $\boldsymbol{L^i = N^i(t)u^i(t)h^i(t)}$ and $\boldsymbol{K^i = v^i(t)k^i(t)}$. The modified utility function is as follows.

$$\boldsymbol{\psi\big(c(t)\big) = \frac{(c^{\beta} c_{im}^{1-\beta})^{1-\sigma}}{1-\sigma}} \qquad (\mathbf{40})$$

The total consumption consists of the domestic produced goods and the imported goods $\boldsymbol{c_{im}}$.

4.1. Case 1: Trade in a less developed country

First a relatively less developed country trades with the world market. A country is relatively less developed if there is a smaller technological development stage compared to the world market, so it is unrelated to developing countries. The development stage has impact on the education sector and the used technology. The initial situation is the autarky situation without trade. The Home country has relatively more physical capital than human capital compared to the world market. That is why the education sector exists in a smaller way. According to that the physical capital-human capital ratio of the considered country is higher than the ratio of the world market.

$$(k(t)/h(t))^{*} > (k(t)/h(t))^{WM}$$

Both regions produce a consumption good which benefits the consumers in the same way. But in the production process the less developed country uses less human capital and substitute it with physical capital. After the reactions through trade the dynamics of growth get priority in the exposition of the model. Relating to the education sector there is a new variation of the countries' dynamics. The less developed country does not spend physical capital on human capital accumulation, but the world market does. Responsible for this different allocation is the development stage of the world market's education sector, which is relatively more developed than the one of the small economy. The less developed country has the same technology for the consumption good as the world market indeed, but the production function of the education sector differs.

These results following the approach of Galor and Mountford (2006). In this work, less developed countries have no incentives to invest additionally physical capital in the education sector. But the trade pattern makes clear, that there are also no incentives to invest in the population size, because the relatively less developed country exports physical capital and there is no good which is produced labor intensive. This raises the question if trade leads to exploiting the resources.

The law of motion for physical capital in an open economy is:

$$\dot{k}(t) = Ak(t)^{\alpha}(u(t)h(t))^{1-\alpha} - c(t) - c(t)_{ex} + p^{*}c(t)_{im} \qquad c(t)_{ex}, c(t)_{im} > 0 \quad (41)$$

The main difference is the openness of the country. The trade flows of the consumption good are considered as $\boldsymbol{c(t)_{ex}}$ and $\boldsymbol{c(t)_{im}}$. The price of the export good is normalised to one. The price of the import good is $\boldsymbol{p}^*$.

Even in this case, the production factors of the consumption good $\boldsymbol{c(t)}$ are human and physical capital. Additionally, there is no more distribution of the human capital $\boldsymbol{v(t)}$.(3) For the production of the handbag sewing machines and tailors are needed. But the education sector uses the tailors to teach new unskilled workers without any use of physical capital.

$$\dot{h}(t) = B\left(1 + \overline{B}(t)\right)\left(1 - u(t)\right)h(t) \tag{42}$$

This distinction shows the different development stages. Because a relatively less developed country is relatively better endowed with physical capital than with human capital, it will be the only production factor of the education sector.

The indirect transfer of knowledge is shown by the parameter of technology $\overline{\boldsymbol{B}}\boldsymbol{(t)}$.

$$\overline{B}(t) = \phi c(t)_{im} \tag{43}$$

The parameter $\boldsymbol{\phi}$ shows the direct knowledge transfer caused by the import of human capital-intensive goods. For the locals it is possible to absorb the imported knowledge to use it for their own production processes, thus $\boldsymbol{0 < \phi(t) < 1}$. The parameter $\overline{\boldsymbol{B}}\boldsymbol{(t)}$ is the diffused knowledge via trade at time $\boldsymbol{t}$, which increases with higher openness of a country. The growth rate of knowledge diffusion depends on the growth rate of the imports.(4)

$$\widehat{(1 + \overline{B})} = \hat{c}_{im} \tag{44}$$

The openness is measured by the volumes of trade flows, which indirectly considers trade barriers. But the households do not observe the knowledge growth by trade thus there is no consideration of these externalities for the optimization problem (Romer, 1986).(5) To solve the maximization problem the Hamiltonian is needed.

$$\begin{aligned}\mathbb{H} = \quad & e^{-\rho t}\frac{(c^{\beta}c_{im}^{1-\beta})^{1-\sigma}}{1-\sigma} \\ & +\gamma_1(Ak^{\alpha}(uh)^{1-\alpha} - c - c_{ex} + p^*c_{im}) \\ & +\gamma_2 B\left(1 + \overline{B}\right)(1-u)h\end{aligned} \tag{45}$$

The shadow price $\boldsymbol{\gamma_{1im}}$ depends on the price of physical capital on the world market, $\boldsymbol{p^*\gamma_1 \,\widehat{=}\, \gamma_{1im}}$. The following conditions describe the equilibrium, which are deduced from the first order conditions of the relatively less developed open economy:

$$\gamma_1 = e^{-\rho t}\beta c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{46}$$

$$\gamma_{1im} = -e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{47}$$

$$\gamma_1 A\alpha k^{\alpha-1}(uh)^{1-\alpha} \overset{!}{=} -\dot{\gamma}_1 \tag{48}$$

$$\gamma_{1im} A\alpha k^{\alpha-1}(uh)^{1-\alpha} \overset{!}{=} -\dot{\gamma}_{1im} \tag{49}$$

$$\gamma_1 A(1-\alpha)k^{\alpha}u^{-\alpha}h^{1-\alpha} = \gamma_2 B(1+\overline{B})h \tag{50}$$

$$\gamma_1 A(1-\alpha)k^{\alpha}u^{1-\alpha}h^{-\alpha} + \gamma_2 B(1+\overline{B})(1-u) = -\dot{\gamma}_2 \tag{51}$$

$$\lim_{t\to\infty} e^{-\rho t}\gamma_1 k = 0; \qquad \lim_{t\to\infty} e^{-\rho t}\gamma_{1im} k = 0; \qquad \lim_{t\to\infty} e^{-\rho t}\gamma_2 h = 0 \tag{52}$$

Model dynamics

The growth rate of domestic produced consumption goods and of the imports result from the restrictions above.

$$\gamma_1 A\alpha k^{\alpha-1}(uh)^{1-\alpha} \overset{!}{=} -\dot{\gamma}_1 \tag{53}$$

$$\gamma_1 A\alpha k^{\alpha-1}(uh)^{1-\alpha} \overset{!}{=} -\dot{\gamma}_1 \tag{54}$$

$$\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha\left(\frac{k}{uh}\right)^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right)$$

$$\boldsymbol{\hat{c}_{im} = \frac{1}{\beta-\sigma\beta+\sigma}\left(A\alpha\left(\frac{k}{uh}\right)^{\alpha-1} - \rho + \hat{c}(\beta-\sigma\beta)\right)}$$

In the steady state both grow with the same rate, hence there is a general growth rate.

$$\hat{c} = \hat{c}_{im}$$

$$\hat{c} = \frac{1}{\sigma}\left(A\alpha\left(\frac{k}{uh}\right)^{\alpha-1} - \rho\right) \tag{55}$$

The growth rate of physical capital depends mainly on the marginal product of physical capital and the optimal relation $\boldsymbol{\chi}$, as well as $\boldsymbol{\chi_{ex}}$ und $\boldsymbol{\chi_{im}}$.

$$\hat{k} = Ak^{\alpha-1}(uh)^{1-\alpha} - \frac{c}{k} - \frac{c_{ex}}{k} + p^*\frac{c_{im}}{k}$$

with $\boldsymbol{\chi = \frac{c}{k}}$, $\boldsymbol{\chi_{ex} = \frac{c_{ex}}{k}}$ and $\boldsymbol{\chi_{im} = \frac{c_{im}}{k}}$ follows

$$\hat{k} = Au^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1} - \chi - \chi_{ex} + p^*\chi_{im} \tag{56}$$

Human capital growth is stimulated by the openness of a country $\boldsymbol{\overline{B}}$, bcause the additional technological knowledge can be invested in the education sector and widen the stock of knowledge.

$$\hat{h} = B(1+\overline{B})(1-u) \tag{57}$$

Moreover, the allocation of human capital is constant over time $\boldsymbol{t}$ in the equilibrium. It shall hold that: $\hat{\boldsymbol{u}} = \boldsymbol{0}$.

$$B(1+\overline{B})\left(\frac{1-\alpha}{\alpha}\right) + B(1+\overline{B})u - \chi = 0 \tag{58}$$

Rearranging this equation, it follows.

$$\chi = \frac{1-\alpha}{\alpha} B(1+\overline{B}) + B(1+\overline{B})u \tag{59}$$

If $\hat{\boldsymbol{v}} = \boldsymbol{0}$ holds, then also $\hat{\boldsymbol{c}} = \hat{\boldsymbol{k}} = \hat{\boldsymbol{h}}$ is true. The marginal product of physical capital is deduced by the condition: $\hat{\boldsymbol{k}} = \hat{\boldsymbol{h}}$.

$$(Ak^{\alpha-1}(uh)^{1-\alpha})^* = \frac{1}{\alpha} B(1+\overline{B}) \tag{60}$$

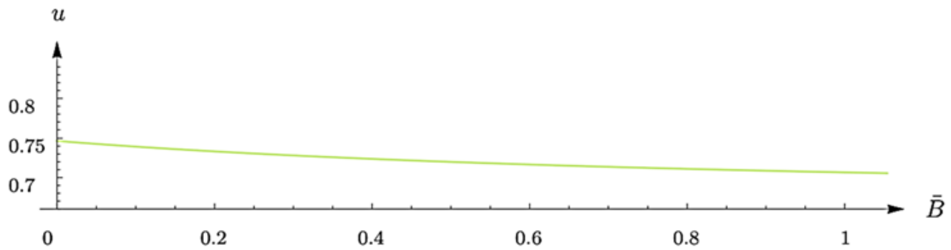
When consumption and physical capital grew with the same rate, then the optimal allocation for physical capital $\boldsymbol{v}^*$ and the the optimal value for the capital-consumption rate $\boldsymbol{\chi}^*$ is calculated by

$$u^* = 1 - \frac{1}{\sigma}\left(1 - \frac{\rho}{B(1+\overline{B})}\right) \tag{61}$$

$$\chi^* = \frac{B(1+\overline{B})}{\alpha} - \frac{B(1+\overline{B}) - \rho}{\sigma} \tag{62}$$

Figure 1 shows the impact of the openness on the optimal allocation of both sectors. The situation of the country could range between closed with $\overline{\boldsymbol{B}} = \boldsymbol{0}$ or free trade with $\overline{\boldsymbol{B}} = \boldsymbol{1}$.

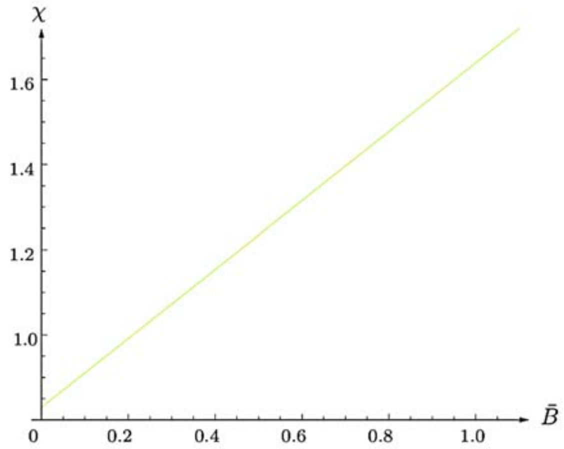
Figure 1. *Dependence of u in the consumption good sector of a relatively less developed country on the openness* $\overline{B}$



Source: Own research.

Technological knowledge can be absorbed more intensive with increasing openness, hence through more imports. Corresponding to that, the education sector benefits from openness because the households will invest more human capital in the education sector than before. With higher openness the allocation of human capital u decreases, which is shown in Figure 1. A similar relationship shows Figure 2. The higher the openness thus international trade of a country the higher is the consumption-capital rate.

Figure 2. *Dependence of χ of a relatively less developed country on the openness* $\overline{B}$



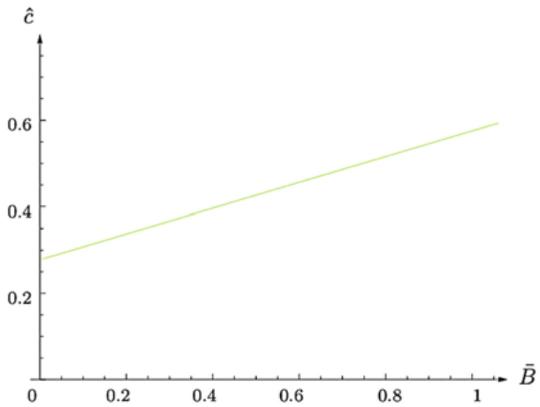
Source: Own research.

Considering these values, the optimal growth rate, the Keynes-Ramsey-Rule, is:

$$\hat{c}^* = \frac{1}{\sigma}\left(\frac{1}{\alpha}B(1+\overline{B}) - \rho\right) \tag{63}$$

Also, to regard is a positive influence of the openness on the economic growth rate. Trade supports an increasing growth rate of a country, shown in Figure 3.

Figure 3. *Dependence of the economic growth rate $\hat{c}$ of a relatively less developed country on the openness* $\overline{B}$



Source: Own research.

4.2. Case 2: Trade in a more developed country

The second case deals with the small country, which is relatively more developed compared to the world market. It is assumed, that the technological development stage of the world market is the average technological progress of the remaining world. Therefore, a more developed country is a very industrialized country which produces high tech manufacturing by using lots of human capital. As well the world market has an absolute advantage in both production factors. The rate of both production factors is the critical item of the development stage.

$$(\boldsymbol{k(t)/h(t)})^{*} < (\boldsymbol{k(t)/h(t)})^{\boldsymbol{WM}}$$

Since the high proportion of human capital in the developed country, the physical capital-human capital ratio of the world market is higher than the one of the home country.

By considering the scenario where a country is relatively more developed in relation to the world market the influence by trade regarding the intertemporal decision for consumption can also be shown. The state of development can be obtained from the relation of physical to human capital. The marginal product in the education sector of a relatively more developed country is also higher than that of a relatively less developed country. Thus, in general it is true that human capital tends to be accumulated at a higher level and both production factors are used in the education sector. In a relatively less developed country not only human capital in the form of teaching staff is needed for the education of tailors but also sewing machines as physical capital. Moreover, it would be possible to use media videos or slides on tablets as teaching resource to spread the knowledge.

A relatively more developed country has therefore relatively more human capital than physical capital in relation to relatively less developed countries or the world market. Thus, the law of motion for physical capital can be stated as:

$$\dot{k}(t) = A(v(t)k(t))^{\alpha}(u(t)h(t))^{1-\alpha} - c(t) - c_{ex}(t) + p^{*}c_{im}(t) \tag{64}$$

Physical capital changes over time. From the produced goods according to $\boldsymbol{y = A(v(t)k(t))^{\alpha}(u(t)h(t))^{1-\alpha}}$ and the valued imported goods $\boldsymbol{p^{*}c_{im}(t)}$ the consumption goods $\boldsymbol{c(t)}$ and exported goods $\boldsymbol{c_{ex}(t)}$ are reallocated to foreign countries. The capital accumulation differs from the relatively less developed country only in the factor $\boldsymbol{v}$ which sets the amount of human capital used in the production sector.

As already stated it is assumed that in a relatively more developed country in addition to human capital also physical capital is employed for education.

$$\dot{h}(t) = B(1+\overline{B})((1-v(t))k(t))^{\eta}((1-u(t))h(t))^{1-\eta} \tag{65}$$

Generally human capital is accumulated in the same manner as by the world market or in closed reference situation considering both capital types. Therefore, human capital develops over time by the input of human capital with $\boldsymbol{((1-u(t))h(t))^{1-\eta}}$ and physical capital with $\boldsymbol{((1-v(t))k(t))^{\eta}}$. Moreover, production parameters $\boldsymbol{B}$ and $\boldsymbol{\overline{B}}$ affecting the education sector.(6)

Through the law of motion the growth rates of both capital types can be obtained:

$$\hat{k} = Av^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} - \frac{c}{k} - \frac{c_{ex}}{k} + p^{*}\frac{c_{im}}{k} \tag{66}$$

$$\hat{h} = B(1+\overline{B})\left[(1-v)\frac{k}{h}\right]^{\eta}(1-u)^{1-\eta} \tag{67}$$

The relation of the physical capital to the amount of consumption can be shortened by $\boldsymbol{\chi} = \frac{\boldsymbol{c}}{\boldsymbol{k}}$, $\boldsymbol{\chi_{ex}} = \frac{\boldsymbol{c_{ex}}}{\boldsymbol{k}}$ and $\boldsymbol{\chi_{im}} = \frac{\boldsymbol{c_{im}}}{\boldsymbol{k}}$. Inserted in (66) it follows:

$$\hat{k} = Av^{\alpha}u^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1} - \chi - \chi_{ex} + p^{*}\chi_{im} \tag{68}$$

Through substitution of $\boldsymbol{x_1} = \frac{\boldsymbol{vk}}{\boldsymbol{uh}}$ and $\boldsymbol{x_2} = \frac{(\boldsymbol{1-v})\boldsymbol{k}}{(\boldsymbol{1-u})\boldsymbol{h}}$ the growth rates can be shortened to

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi - \chi_{ex} + p^{*}\chi_{im} \tag{69}$$

$$\hat{h} = B\left(1+\overline{B}\right)x_2^{\eta}(1-u) \tag{70}$$

Again, the maximization problem can be solved using the Hamiltonian approach. Finding the optimal path of consumption which maximizes the individual lifetime utility in a relatively more developed country which is free for trade.

$$\begin{aligned} \mathbb{H} = \quad & e^{-\rho t}\frac{(c^{\beta}c_{im}^{1-\beta})^{1-\sigma}}{1-\sigma} \\ & +\gamma_1(A(vk)^{\alpha}(uh)^{1-\alpha} - c - c_{ex} + p^{*}c_{im}) \\ & +\gamma_2 B(1+\overline{B})[(1-v)k]^{\eta}[(1-u)h]^{1-\eta} \end{aligned} \tag{71}$$

The first order conditions of the relatively more developed country are similar to the ones described before. The similarity to the reference situation is given because human capital is accumulated in the same way. The growth path is also similar to relatively less developed countries since both are open economies and the parameter of openness affects the growth path. In summary a relatively more developed country is in the steady-state if the following restrictions are true:

$$\gamma_1 = e^{-\rho t}\beta c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{72}$$

$$\gamma_{1im} = -e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{73}$$

$$\gamma_1 A\alpha v^{\alpha-1}k^{\alpha}(uh)^{1-\alpha} = \gamma_2 B(1+\overline{B})\eta(1-v)^{\eta-1}k^{\eta}[(1-u)h]^{1-\eta} \tag{74}$$

$$\gamma_1 A\alpha v^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} + \gamma_2 B(1+\overline{B})(1-v)^{\eta}k^{\eta-1}\eta[(1-u)h]^{1-\eta} = -\dot{\gamma}_1 \tag{75}$$

$$\gamma_{1im}A\alpha v^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} + \gamma_2 B(1+\overline{B})[h(1-u)]^{1-\eta}\eta(1-v)^{\eta}k^{\eta-1} = -\dot{\gamma}_{1im} \tag{76}$$

$$\gamma_1 A(1-\alpha)(vk)^{\alpha}u^{-\alpha}h^{1-\alpha} = \gamma_2 B(1+\overline{B})(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta}h^{1-\eta} \tag{77}$$

$$\gamma_1 A(1-\alpha)(vk)^{\alpha}u^{1-\alpha}h^{-\alpha} + \gamma_2 B(1+\overline{B})(1-\eta)[(1-v)k]^{\eta}(1-u)^{1-\eta}h^{-\eta} = -\dot{\gamma}_2 \tag{78}$$

$$\lim_{t\to\infty} e^{-\rho t}\gamma_1 k = 0; \qquad \lim_{t\to\infty} e^{-\rho t}\gamma_{1im} k = 0; \qquad \lim_{t\to\infty} e^{-\rho t}\gamma_2 h = 0 \tag{79}$$

As shown in equation (72) the shadow price of an additional unit capital is described. The future marginal lifetime utility has to be equal to the current lifetime utility, which results from an additional consumed unit. The interrelation is true for the imported good $\boldsymbol{c_{im}}$ characterized in equation (73).

Constraint (74) and equation (77) ensure that an economy is in the steady-state when the marginal products valued with the shadow price are equally.

The remaining constraints (75), (76) and (78) state the respective discount rate of the shadow prices. Those with trade in relative more developed countries are higher than in the reference situation without trade. The utility of an additional capital unit is therefore higher and has to be amortized over the lifetime.

Model dynamics

These conditions lead to the Keynes-Ramsey-Rules, for the growth rate of the consumption good and the imports.

$$\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha x_1^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right) \tag{80}$$

$$\hat{c}_{im} = \frac{1}{\beta-\sigma\beta+\sigma}\left(A\alpha x_1^{\alpha-1} - \rho + \hat{c}(\beta-\sigma\beta)\right) \tag{81}$$

Both growth rates are entailing of each other. The equivalence of both marginal productivities of both sectors is followed by equation (74) and (75).

$$\frac{1-\alpha}{\alpha}x_1 = \frac{1-\eta}{\eta}x_2 \tag{82}$$

The marginal productivities of human or physical capital in the consumption good sector correlate and are independent of trade. The following equation shows that the growth rate of the shadow price of physical capital is equal to the marginal product of physical capital.

$$B\left(1+\overline{B}\right)(1-\eta)x_2^{\eta} = -\rho - \hat{c}(\beta-1-\sigma\beta) - \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{83}$$

The optimal growth path of a relatively more developed country is described by:

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi - \chi_{ex} + p^*\chi_{im} \tag{84}$$

$$\hat{h} = B\left(1+\overline{B}\right)x_2^{\eta}(1-u) \tag{85}$$

$$\frac{x_1(1-\alpha)}{\alpha} = \frac{x_2(1-\eta)}{\eta} \tag{86}$$

$$\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha x_1^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right) \tag{87}$$

$$\hat{c}_{im} = \frac{1}{\beta-\sigma\beta+\sigma}\left(A\alpha x_1^{\alpha-1} - \rho + \hat{c}(\beta-\sigma\beta)\right) \tag{88}$$

$$B\left(1+\overline{B}\right)(1-\eta)x_2^{\eta} = -\rho - \hat{c}(\beta-1-\sigma\beta) - \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{89}$$

The values of the equilibrium whose growth rate is constant of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are:

$$x_2^* = \left(\frac{\rho + \sigma\hat{c}}{B(1+\overline{B})(1-\eta)}\right)^{1/\eta} \tag{90}$$

$$x_1^* = \frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho + \sigma\hat{c}}{B(1+\overline{B})(1-\eta)}\right)^{1/\eta} \tag{91}$$

Assuming the country is in the equilibrium of foreign trade, it holds that: $\hat{\boldsymbol{c}} = \hat{\boldsymbol{c}}_{\boldsymbol{im}}$.

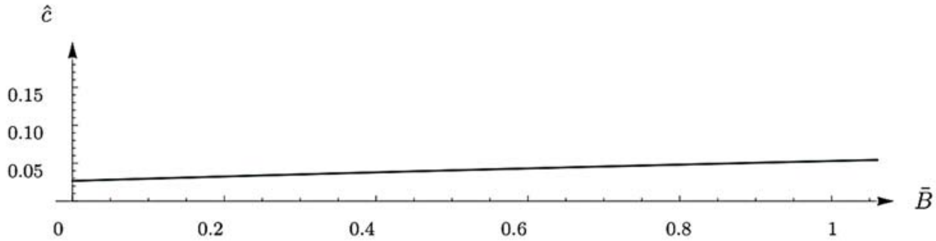
$$\hat{c} = \frac{1}{\sigma}(A\alpha x_1^{\alpha-1} - \rho) \tag{92}$$

This yields a balanced growth path for the open economy, depending on $\overline{\boldsymbol{B}}$.

$$\hat{c}^* = \frac{1}{\sigma}\left(\left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}(B(1+\overline{B}))^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}} - \rho\right) \tag{93}$$

The more open an economy, the more knowledge could be absorbed by importing human capital goods and the higher the growth rate.(7) This relationship is shown in Figure 4.

Figure 4. *Dependence of the growth rate $\hat{c}$ of a relatively more developed country on the openness $\overline{B}$*



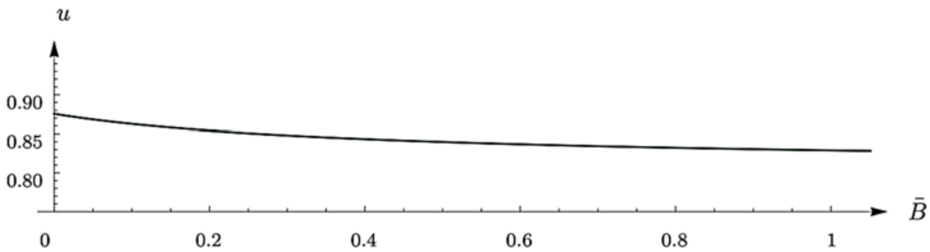
Source: Own research.

Because the equilibrium holds $\hat{c} = \hat{h}$, the optimal value u^* is:

$$u^* = \frac{\sigma\overline{M} - (1-\eta)(\overline{M} - \rho)}{\sigma\overline{M}} \tag{94}$$

With increasing knowledge transfer by openness $\overline{\boldsymbol{B}}$ the less capital is invested in the consumption good sector, caused by $(\boldsymbol{1} - \boldsymbol{u})$, as shown in Figure 5.

Figure 5. *Dependence of the amount of human capital u in the consumption good sector of a relatively more developed country on the openness $\overline{B}$*



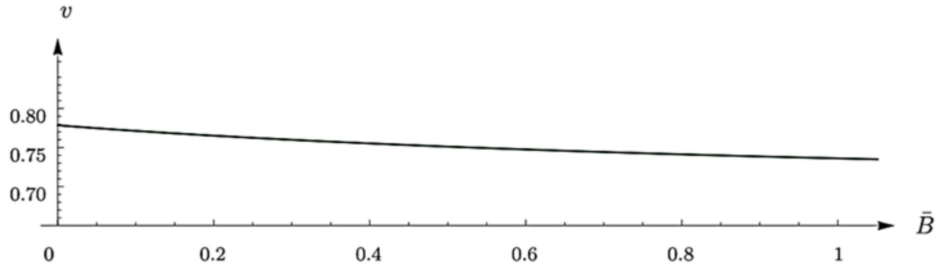
Source: Own research.

Caused by international trade both factors of production get rearranged and allocated in a new way. The optimal amount of physical capital v in the equilibrium is:

$$v^* = \frac{\alpha(1-\eta)\left(\frac{\overline{M}}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(\overline{M}\sigma-(1-\eta)(\overline{M}-\rho)\right)}{(1-\alpha)\eta\overline{M}\sigma\left(\frac{\alpha(1-\eta)\left(\frac{\overline{M}}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(\overline{M}\sigma-(1-\eta)(\overline{M}-\rho)\right)}{(1-\alpha)\eta\overline{M}\sigma}+\left(\frac{\overline{M}}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(1-\frac{\overline{M}\sigma-(1-\eta)(\overline{M}-\rho)}{\overline{M}\sigma}\right)\right)} \tag{95}$$

Figure 6 shows the inverse relationship between openness and the investment of physical capital in the consumption good sector. Furthermore, a constant capital-consumption good rate $\boldsymbol{\chi}$ is necessary for the balanced equilibrium.

Figure 6. *Dependence of the amount of physical capital v of the consumption good sector of a relative more developed country on the openness* $\overline{B}$

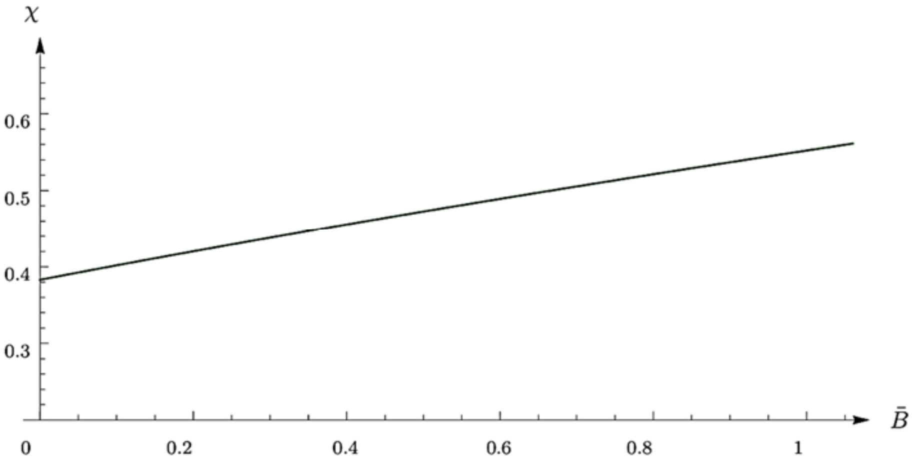


Source: Own research.

It holds, if $\hat{\boldsymbol{c}} = \hat{\boldsymbol{k}}$. The dependence of this rate on the openness is shown in Figure 7.

$$\chi^* = \frac{1}{\sigma}\left(\frac{A\alpha\sigma[-\eta\rho+\overline{M}(\eta+\sigma-1)+\rho]\left(\frac{\alpha(\eta-1)\left(\frac{\overline{M}}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}}{(\alpha-1)\eta}\right)^{\alpha-1}}{\rho(\alpha-\eta)+\overline{M}(\alpha(\sigma-1)+\eta)}-\overline{M}+\rho\right) \tag{96}$$

Figure 7. *Dependence of the capital-consumption good rate $\boldsymbol{\chi}$ of a relatively more developed country on the openness* $\overline{B}$



Source: Own research.

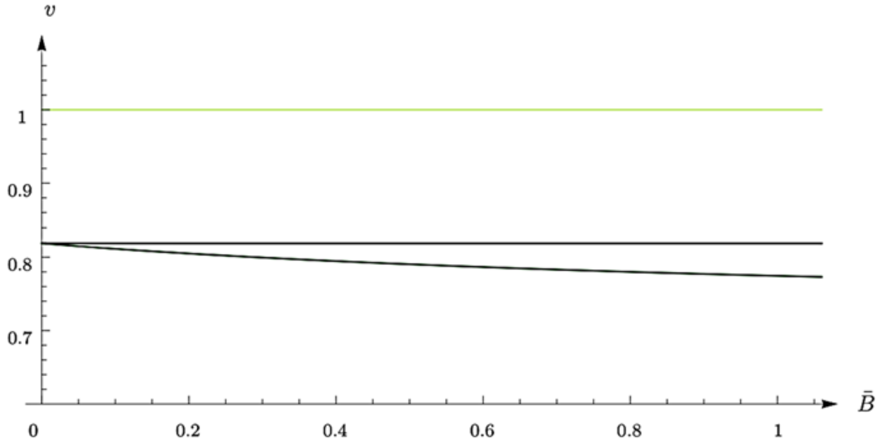
5. Conclusion

This paper has proved two extensions of the Uzawa Lucas framework. The focus is on the introduction of trade with two accumulating factors for a small economy. Two different development stages of this country are analyzed. The first case is about a relatively less developed country with a higher physical capital-human capital ratio $\boldsymbol{h(t)/k(t)}$ than the world market. The second situation observes a more developed country compared to the world market with a smaller $\boldsymbol{h(t)/k(t)}$ ratio. The less developed country has no physical capital in the education sector whereas the more developed country uses physical capital as an additional input in human capital accumulation.

The relatively less developed country improves its economic situation. Trade induces a higher human capital accumulation. This country benefits from the import of the consumption good which is produced with relatively more human capital at the world market than at the domestic market.

To illustrate the results of both cases, they are compared to the autarky situation. Looking at the amount of physical capital which is invested in the consumption good sector $\boldsymbol{v}$, there are indirect results for the education sector derivable, because the amount of physical capital of the education sector is described with $\boldsymbol{(1-v)}$. The comparison of the development stages is shown in Figure 8. In a relatively less developed country $\boldsymbol{v}$ is decreasing with raising openness. But the education sector benefits from trade compared to autarky. Now the education sector uses more physical capital than before trade.

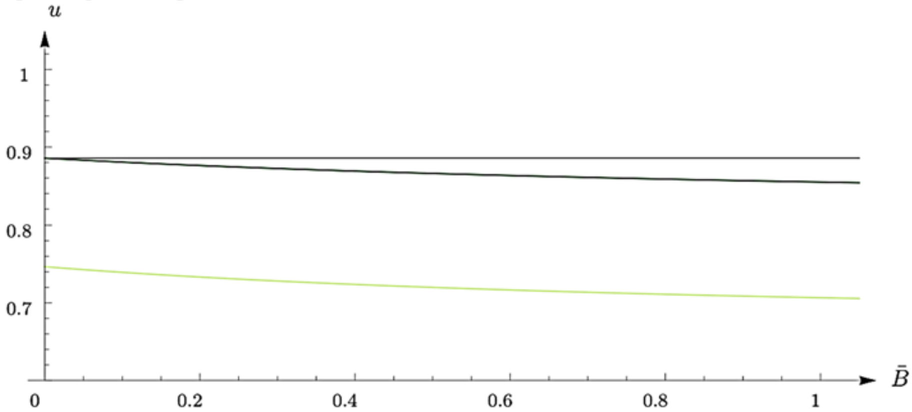
Figure 8. *Variance of amount of physical capital v in the consumption sector on different development stages depending on the openness* $\overline{B}$



Source: Own research.

The level of the control variable $\boldsymbol{v}$ in relatively less developed countries is higher and constant because $\boldsymbol{v = 1}$. The hole physical capital stock is used in the consumption good sector. This leads to a smaller investment of human capital in the consumption sector, thus the investment of human capital in the education sector is relatively higher than in a relatively more developed country. This is shown in Figure 9.

Figure 9. *Variance of the amount of human capital u in the consumption sector on different development stages depending on the openness* $\overline{B}$



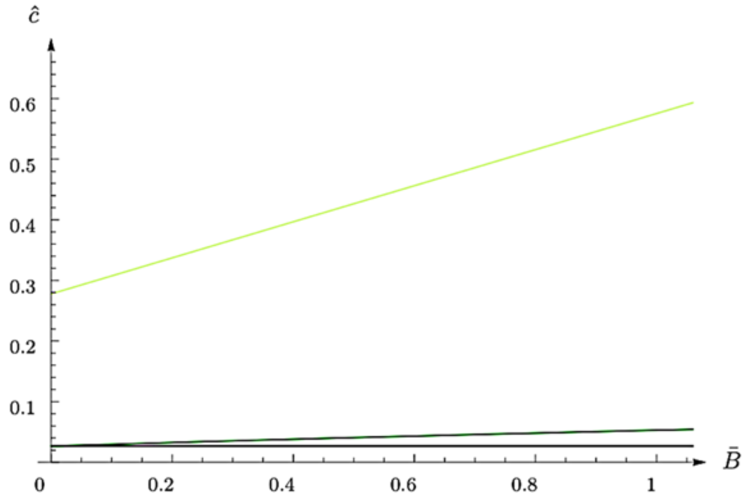
Source: Own research.

Independent of the state of development there will always be an allocation of human capital to both sectors. Induced by international trade the amount of human capital of the consumption sector u decreases. According to that the fraction of the education sector $(1-u)$ increases. But there are also level differences observed. A relatively less developed country invests on average ca. 25% of the human capital in the education sector, this share increases with the openness of a country. The households of relative more developed countries decide to spend less human capital in the consumption good sector but more in the education sector. Compared to the autarky situation in both development stages the education sector benefits from the openness.

International trade created an incentive to invest more in the education sector, independent of the state of development.

The openness of a country intensifies the knowledge transfer and the accompanying technological diffusion. The model shows a catching up process for relatively less developed countries relating to the human capital stock caused by international trade. This leads finally to a higher growth rate which converges to the world market, shown in Figure 10.

Figure 10. *Comparison of the growth rates* $\hat{\boldsymbol{c}}$ *on different development stages depending on the openness* $\overline{B}$



Source: Own research.

The reference situation of a closed economy is very similar to the relative more developed country considering the structure of productions with $v > 0$. That is why they start at the same level. Therefore, the economic growth of both cases is the same with $\overline{B} = 0$. The open and relative more developed country has an increasing growth rate induced by the openness. The relative less developed country has an obviously higher starting point and is definitely increasing with increasing openness. This indicates the catching up process of the relatively less developed country. The shift of the marginal product of physical capital in relative less developed countries is higher than in more developed countries and thus leads to a higher growth rate. Also, the lack of knowledge is much bigger in relative less developed countries. To sum up it is shown that independent of the development stage openness and international trade stimulates the economic growth rate.

Notes

(1) A teacher can be every skilled worker who already produced a few bags. Per definition human capital is not necessarily linked to workers. There exists a stock of knowledge which is applied by the workers.

(2) Replacing $\chi = \frac{c}{k}$ two kinds of growth rate notations follow for physical capital.

(3) To be more precise: $\upsilon = 1$.

(4) For simplicity the dependence on time t is neglected again.

(5) That is why $\overline{B}$ is exogenous in this model.

(6) The openness factor complies as in (43).

(7) To shorten the equation: $\overline{M} = \left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}(B(1+\overline{B}))^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}}$

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A. Mathematical Appendix

A.1. Autarky

The laws of motion for physical and human capital are given by

$$\dot{k}(t) = A(v(t)k(t))^{\alpha}(u(t)h(t))^{1-\alpha} - c(t) \tag{97}$$

$$\dot{\boldsymbol{h}}(\boldsymbol{t}) = \boldsymbol{B}((\boldsymbol{1}-\boldsymbol{v}(\boldsymbol{t}))\boldsymbol{k}(\boldsymbol{t}))^{\boldsymbol{\eta}}((\boldsymbol{1}-\boldsymbol{u}(\boldsymbol{t}))\boldsymbol{h}(\boldsymbol{t}))^{\boldsymbol{1}-\boldsymbol{\eta}} \tag{98}$$

For reasons of clarity the dependence of time $\boldsymbol{t}$ has been obmitted. The growth rate of physical capital is:

$$\hat{k} = Av^{\alpha}k^{\alpha-1}(uh)^{1-\alpha} - \frac{c}{k}$$

with $\boldsymbol{\chi} = \frac{\boldsymbol{c}}{\boldsymbol{k}}$ follows

$$\hat{k} = Av^{\alpha}u^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1} - \chi \tag{99}$$

By substituting $\boldsymbol{x_1} = \frac{\boldsymbol{vk}}{\boldsymbol{uh}}$ a simplified form of the growths rate is:

$$\boxed{\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi} \tag{100}$$

The human capital growths as followed:

$$\hat{h} = B\left[(1-v)\frac{k}{h}\right]^{\eta}(1-u)^{1-\eta} \tag{101}$$

Also the growth rate can be written in a simpler way by the substitution of $\boldsymbol{x_2} = \frac{(\boldsymbol{1}-\boldsymbol{v})\boldsymbol{k}}{(\boldsymbol{1}-\boldsymbol{u})\boldsymbol{h}}$.

$$\boxed{\hat{h} = Bx_2^{\eta}(1-u)} \tag{102}$$

The household solves the maximization problem with the Hamiltonian approach.

$$\mathbb{H} = \begin{array}{l} \boldsymbol{e}^{-\boldsymbol{\rho t}}\dfrac{\boldsymbol{c}^{\boldsymbol{1}-\boldsymbol{\sigma}}}{\boldsymbol{1}-\boldsymbol{\sigma}} \\ +\boldsymbol{\gamma_1}(\boldsymbol{A}(\boldsymbol{vk})^{\boldsymbol{\alpha}}(\boldsymbol{uh})^{\boldsymbol{1}-\boldsymbol{\alpha}} - \boldsymbol{c}) \\ +\boldsymbol{\gamma_2}\boldsymbol{B}[(\boldsymbol{1}-\boldsymbol{v})\boldsymbol{k}]^{\boldsymbol{\eta}}[(\boldsymbol{1}-\boldsymbol{u})\boldsymbol{h}]^{\boldsymbol{1}-\boldsymbol{\eta}} \end{array} \tag{103}$$

The first order conditions are given by:

$$\frac{\partial \mathbb{H}}{\partial c} \overset{!}{=} 0 \tag{104}$$

$$\frac{\partial \mathbb{H}}{\partial v} \overset{!}{=} 0 \tag{105}$$

$$\frac{\partial \mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma}_1 \tag{106}$$

$$\frac{\partial \mathbb{H}}{\partial u} \overset{!}{=} 0 \tag{107}$$

$$\frac{\partial \mathbb{H}}{\partial h} \overset{!}{=} -\dot{\gamma}_2 \tag{108}$$

The correlation of equation (104) yields:

$\partial H/\partial c \overset{!}{=} 0$

$$\boldsymbol{\gamma_1 = e^{-\rho t} c^{-\sigma}} \tag{109}$$

The law of motion of the shadow price $\boldsymbol{\gamma_1}$ is needed for the Keynes Ramsey Rule. The derivative of equation (109) with respect to time is:

$$\frac{\partial \gamma_1}{\partial t} = \dot{\gamma}_1 \tag{110}$$

$$\boldsymbol{\dot{\gamma}_1 = -e^{-\rho t} c^{-\sigma}(\rho + \sigma \hat{c}) = -\gamma_1(\rho + \sigma \hat{c})} \tag{111}$$

Combining this equation with (109) the growth rate of the shadow price of good one in a closed economy result:

$$\widehat{\gamma_1} = -\rho - \sigma \hat{c} \tag{112}$$

Condition (105) determines the optimal allocation of physical capital with maximal life time utility.

$\partial H/\partial c \overset{!}{=} 0$

$$\boldsymbol{\gamma_1 A \alpha v^{\alpha-1} k^{\alpha} (uh)^{1-\alpha} = \gamma_2 B \eta (1-v)^{\eta-1} k^{\eta} [(1-u)h]^{1-\eta}} \tag{113}$$

Then the relation of the shadow prices is given by:

$$\frac{\gamma_2}{\gamma_1} = \frac{A\alpha \left(\frac{vk}{uh}\right)^{\alpha-1}}{B\eta \left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} = \frac{A\alpha x_1^{\alpha-1}}{B\eta x_2^{\eta-1}} \tag{114}$$

$$\boldsymbol{\gamma_2 = \gamma_1 \frac{A\alpha \left(\frac{vk}{uh}\right)^{\alpha-1}}{B\eta \left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} \Leftrightarrow \gamma_1 = \gamma_2 \frac{B\eta \left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}}{A\alpha \left(\frac{vk}{uh}\right)^{\alpha-1}}} \tag{115}$$

The growth rate of the shadow price of good one is deduced from equation (106).

$\partial H/\partial k \overset{!}{=} -\dot{\gamma}_1$

$$\boldsymbol{A\alpha v^{\alpha} k^{\alpha-1}(uh)^{1-\alpha} + \frac{\gamma_2}{\gamma_1} B\eta(1-v)^{\eta} k^{\eta-1}[(1-u)h]^{1-\eta} = -\hat{\gamma}_1} \quad (116)$$

By using the ratio of the shadow prices $\boldsymbol{\gamma_2/\gamma_1}$ of equation (114) it arises:

$$A\alpha v^{\alpha} u^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1} + \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} B\eta(1-v)^{\eta}k^{\eta-1}[(1-u)h]^{1-\eta} = -\hat{\gamma}_1$$

$$\boldsymbol{\hat{\gamma}_1 = -A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1} \quad \Leftrightarrow \quad \hat{\gamma}_1 = -A\alpha x_1^{\alpha-1}} \quad (117)$$

The Keynes-Ramsey-Rule results by the equations (116) with $\boldsymbol{\gamma_2}$ by (115) and $\boldsymbol{\dot{\gamma}_1}$ by (111).

$$\gamma_1 A\alpha v^{\alpha}\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} + \gamma_1 \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} B\eta(1-v)^{\eta}k^{\eta-1}\left[h(1-u)\right]^{1-\eta} \overset{!}{=} \gamma_1(\rho+\sigma\hat{c}) \quad (118)$$

$$\boxed{\boldsymbol{\hat{c} = \frac{1}{\sigma}\left(A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1} - \rho\right)}} \quad (119)$$

Equation (107) determines the optimal allocation of human capital.

$\partial H/\partial u \overset{!}{=} 0$

$$\boldsymbol{\gamma_1 A(1-\alpha)(vk)^{\alpha} h^{1-\alpha} u^{-\alpha} = \gamma_2 B(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta}h^{1-\eta}} \quad (120)$$

$$\boldsymbol{\frac{\gamma_2}{\gamma_1} = \frac{A(1-\alpha)(vk)^{\alpha}h^{1-\alpha}u^{-\alpha}}{B(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta}h^{1-\eta}}} \quad (121)$$

A ratio of both shadow prices results again.

$$= \frac{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}}{B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}} = \frac{A(1-\alpha)x_1^{\alpha}}{B(1-\eta)x_2^{\eta}} \quad (122)$$

$$\boldsymbol{\gamma_1 = \gamma_2 \frac{B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}}{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}} \Leftrightarrow \gamma_2 = \gamma_1 \frac{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}}{B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}}}$$

$$\boldsymbol{= \gamma_1 \frac{A(1-\alpha)x_1^{\alpha}}{B(1-\eta)x_2^{\eta}}} \qquad \mathbf{(123)}$$

Thus, the growth rate of the shadow price of good two is:

$$\hat{\gamma}_2 = \hat{\gamma}_1 + \alpha\hat{x}_1 - \eta\hat{x}_2 \qquad (124)$$

The ratios of the shadow prices by (114) and (122) are equated.

$$\frac{A\alpha x_1^{\alpha-1}}{B\eta x_2^{\eta-1}} = \frac{A(1-\alpha)x_1^{\alpha}}{B(1-\eta)x_2^{\eta}} \qquad (125)$$

$$\boxed{\boldsymbol{\frac{1-\alpha}{\alpha}x_1 = \frac{1-\eta}{\eta}x_2}} \qquad \mathbf{(126)}$$

The last first order condition (108) is calculated as:

$$\partial H/\partial h \stackrel{!}{=} -\dot{\gamma}_2$$

$$\boldsymbol{\gamma_1 A(1-\alpha)(vk)^{\alpha}u^{1-\alpha}h^{-\alpha} + \gamma_2 B(1-\eta)[(1-v)k]^{\eta}(1-u)^{1-\eta}h^{-\eta} \stackrel{!}{=} -\dot{\gamma}_2} \qquad \mathbf{(127)}$$

The shadow price $\boldsymbol{\gamma_1}$ will be substituted by (123).

$$\gamma_2 \frac{B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}}{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}} A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha} u + \gamma_2 B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}(1-u) = -\dot{\gamma}_2$$

$$\boldsymbol{\hat{\gamma}_2 = -B(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta} \Leftrightarrow \hat{\gamma}_2 = -B(1-\eta)x_2^{\eta}} \qquad \mathbf{(128)}$$

Combining equation (128) and (124) it yields:

$$\hat{\gamma}_1 + \alpha\hat{x}_1 - \eta\hat{x}_2 = -B(1-\eta)x_2^{\eta} \qquad (129)$$

The growth rate of the shadow price of good one by equation (112) is used.

$$\boxed{B(1-\eta)x_2^{\eta} = \rho - \sigma\hat{c} - \alpha\hat{x}_1 + \eta\hat{x}_2} \qquad (130)$$

This leads to the following system of equations, which describes the equilibrium.

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi \tag{131}$$

$$\hat{h} = Bx_2^{\eta}(1-u) \tag{132}$$

$$\frac{x_1(1-\alpha)}{\alpha} = \frac{x_2(1-\eta)}{\eta} \tag{133}$$

$$\hat{c} = \frac{1}{\sigma}(A\alpha x_1^{\alpha-1} - \rho) \tag{134}$$

$$B(1-\eta)x_2^{\eta} = \rho - \sigma\hat{c} - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{135}$$

The equilibrium condition (133) shows the reactions of the relations $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in the long run, by contributing the growth rate of $\boldsymbol{x_1}$. Both grew with the same rate.

$$\hat{x}_1 = \hat{x}_2 = 0 \tag{136}$$

Additionally, in the steady state $\boldsymbol{\hat{c}} = \boldsymbol{\hat{k}} = \boldsymbol{\hat{h}}$ is valid. By solving condition (135) $\boldsymbol{x_2^*}$ yields:

$$B(1-\eta)x_2^{\eta} = \rho - \sigma\hat{c} \tag{137}$$

$$\boldsymbol{x_2^* = \left(\frac{\rho + \sigma\hat{c}}{B(1-\eta)}\right)^{1/\eta}} \tag{\textbf{138}}$$

In the same way results $\boldsymbol{x_1^*}$ by condition (133) and by substituting $\boldsymbol{x_2^*}$.

$$\frac{1-\alpha}{\alpha}x_1 = \frac{1-\eta}{\eta}\left(\frac{\rho + \sigma\hat{c}}{B(1-\eta)}\right)^{1/\eta} \tag{139}$$

$$\boldsymbol{x_1^* = \frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho + \sigma\hat{c}}{B(1-\eta)}\right)^{\frac{1}{\eta}}} \tag{\textbf{140}}$$

By equation (134) follows the optimal balanced growth rate:

$$\hat{c} = \frac{1}{\sigma}\left(A\alpha\left[\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho + \sigma\hat{c}}{B1-\eta)}\right)^{\frac{1}{\eta}}\right]^{\alpha-1} - \rho\right) \tag{141}$$

$$\boxed{\boldsymbol{\hat{c}^* = \frac{1}{\sigma}\left(\left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}(B)^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}} - \rho\right)}} \tag{\textbf{142}}$$

For the following steps the marginal product of physical capital is shortened with the variable $\boldsymbol{M = \left[A^{\eta}\alpha^{\alpha\eta}(1-\eta)^{(1-\eta)(1-\alpha)}(\eta(1-\alpha))^{\eta(1-\alpha)}(B)^{1-\alpha}\right]^{\frac{1}{1+\eta-\alpha}}}$. The optimal allocation of human capital $\boldsymbol{u^*}$ is contributed by $\boldsymbol{\hat{c}} = \boldsymbol{\hat{h}}$ in accordance with (132) and considering $\boldsymbol{x_2^*}$ and $\boldsymbol{\hat{c}^*}$.

$$\frac{1}{\sigma}(M-\rho) = B\left(\left(\frac{\rho+\sigma\frac{1}{\sigma}(M-\rho)}{B(1-\eta)}\right)^{\frac{1}{\eta}}\right)^{\alpha}(1-u) \tag{143}$$

$$\boxed{\boldsymbol{u^{*}} = \frac{\boldsymbol{\sigma M-(1-\eta)(M-\rho)}}{\boldsymbol{\sigma M}}} \tag{144}$$

By the steady state condition $\hat{\boldsymbol{c}} = \hat{\boldsymbol{k}}$ yields the optimal relation of physical capital and human capital using $\boldsymbol{x_1^*}$, $\boldsymbol{x_2^*}$ and $\hat{\boldsymbol{c}}^*$ in equation (131).

$$\frac{1}{\sigma}(M-\rho) = A\left(\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\sigma\frac{1}{\sigma}(M-\rho)}{B(1-\eta)}\right)^{\frac{1}{\eta}}\right)^{\alpha}\frac{h}{k}\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}-\chi$$

$$\boxed{\boldsymbol{\chi^{*}} = \frac{\boldsymbol{1}}{\boldsymbol{\sigma}}\left(\frac{\boldsymbol{A\alpha\sigma[-\eta\rho+M(\eta+\sigma-1)+\rho]}\left(\frac{\boldsymbol{\alpha(\eta-1)}\left(\frac{\boldsymbol{M}}{\boldsymbol{B(1-\eta)}}\right)^{\boldsymbol{1/\eta}}}{\boldsymbol{(\alpha-1)\eta}}\right)^{\boldsymbol{\alpha-1}}}{\boldsymbol{\rho(\alpha-\eta)+M(\alpha(\sigma-1)+\eta)}}-\boldsymbol{M}+\boldsymbol{\rho}\right)} \tag{145}$$

The general equation $\boldsymbol{v} = \frac{\boldsymbol{vk}}{\boldsymbol{uh}}\frac{\boldsymbol{uh}}{\boldsymbol{k}}$ is solved with $\boldsymbol{x_1^*}$, $\boldsymbol{x_2^*}$ and $\frac{\boldsymbol{k}}{\boldsymbol{h}} = \boldsymbol{ux_1}+(\boldsymbol{1}-\boldsymbol{u})\boldsymbol{x_2}$.

$$v = \frac{\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1-\eta)}\right)^{\frac{1}{\eta}}\frac{M\sigma-(1-\eta)(M-\rho)}{\sigma M}}{\frac{1}{\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1-\eta)}\right)^{\frac{1}{\eta}}+\left(1-\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}\right)\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1-\eta)}\right)^{\frac{1}{\eta}}}} \tag{146}$$

The optimal allocation of physical capital $\boldsymbol{v}^*$ is:

$$v^* = \frac{\alpha(1-\eta)\left(\frac{M}{BC(1-\eta)}\right)^{\frac{1}{\eta}}(M\sigma-(1-\eta)(M-\rho))}{(1-\alpha)\eta M\sigma\left(\frac{\alpha(1-\eta)\left(\frac{M}{BC(1-\eta)}\right)^{\frac{1}{\eta}}(M\sigma-(1-\eta)(M-\rho))}{(1-\alpha)\eta M\sigma}+\left(\frac{M}{BC(1-\eta)}\right)^{\frac{1}{\eta}}\left(1-\frac{M\sigma-(1-\eta)(M-\rho)}{M\sigma}\right)\right)} \tag{147}$$

A.2. Open relatively less developed country

The laws of motion are given by:

$$\dot{k}(t) = Ak(t)^{\alpha}(u(t)h(t))^{1-\alpha} - c(t) - c_{ex}(t) + p^* c_{im}(t) \tag{148}$$

$$\boldsymbol{\dot{h}(t) = B\overline{B}\big(1-u(t)\big)h(t)} \tag{149}$$

Neglecting the dependence on the time $\boldsymbol{t}$, the growth rate of physical capital is:

$$\hat{k} = Ak^{\alpha-1}(uh)^{1-\alpha} - \frac{c}{k} - \frac{c_{ex}}{k} + p^*\frac{c_{im}}{k}$$

with $\boldsymbol{\chi} = \frac{\boldsymbol{c}}{\boldsymbol{k}}$, $\boldsymbol{\chi_{ex}} = \frac{\boldsymbol{c_{ex}}}{\boldsymbol{k}}$ as well as $\boldsymbol{\chi_{im}} = \frac{\boldsymbol{c_{im}}}{\boldsymbol{k}}$ follows

$$\hat{k} = Au^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1} - \chi - \chi_{ex} + p^*\chi_{im} \tag{150}$$

The growth rate of human capital is:

$$\hat{h} = B\overline{B}(1-u) \tag{151}$$

The optimization problem is solved with the Hamiltonian again.

$$\begin{aligned}\mathbb{H} = \quad & e^{-\rho t}\frac{(c^{\beta}c_{im}^{1-\beta})^{1-\sigma}}{1-\sigma} \\ & +\gamma_1(Ak^{\alpha}(uh)^{1-\alpha} - c - c_{ex} + p^* c_{im}) \\ & +\gamma_2 B\overline{B}(1-u)h\end{aligned} \tag{152}$$

With the following first order conditions:

$$\frac{\partial\mathbb{H}}{\partial c} \overset{!}{=} 0 \tag{153}$$

$$\frac{\partial\mathbb{H}}{\partial c_{im}} \overset{!}{=} 0 \tag{154}$$

$$\frac{\partial \mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma}_1 \tag{155}$$

$$\frac{\partial \mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma}_{1im} \tag{156}$$

$$\frac{\partial \mathbb{H}}{\partial u} \overset{!}{=} 0 \tag{157}$$

$$\frac{\partial \mathbb{H}}{\partial h} \overset{!}{=} -\dot{\gamma}_2 \tag{158}$$

From equation (153) yields:

$\partial H/\partial c \overset{!}{=} 0$

$$\gamma_1 = e^{-\rho t}\beta c^{\beta-1} c_{im}^{1-\beta} (c^{\beta} c_{im}^{1-\beta})^{-\sigma} \tag{159}$$

The derivation with respect to time $\boldsymbol{t}$ by equation (159) is:

$$\frac{\partial \gamma_1}{\partial t} = \dot{\gamma}_1 \tag{160}$$

$$\begin{aligned}\boldsymbol{\dot{\gamma}_1 =} \quad & \boldsymbol{-e^{-\rho t}\beta\rho c^{\beta-1} c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} + e^{-\rho t}\beta(\beta-1)c^{\beta-2}\dot{c}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}} \\ & \boldsymbol{+e^{-\rho t}\beta c^{\beta-1}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}(1-\beta)c_{im}^{1-\beta-1}\dot{c}_{im}} \\ & \boldsymbol{-e^{-\rho t}\beta\sigma c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-1-\sigma}(c_{im}^{1-\beta}\beta c^{\beta-1}\dot{c} + c^{\beta}c_{im}^{1-\beta-1}(1-\beta)\dot{c}_{im})}\end{aligned}$$

$$\dot{\gamma}_1 = e^{-\rho t}\beta c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}[-\rho + \hat{c}(\beta-1-\sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)] \tag{161}$$

Simplified it follows

$$\dot{\gamma}_1 = \gamma_1[-\rho + \hat{c}(\beta-1-\sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)]$$

and the growth rate of the shadow price of good one arises:

$$\widehat{\gamma_1} = -\rho + \hat{c}(\beta-1-\sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) \tag{162}$$

The consumption of foreign produced and imported goods affects the utility function and follows by equation (154).

$\partial H/\partial c_{im} \overset{!}{=} 0$

$$p^*\gamma_1 \mathrel{\widehat{=}} \gamma_{1im} = -e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{163}$$

The shadow price of the import goods is the valued shadow price of good one $\boldsymbol{\gamma_1}$. Its derivative with respect to time is:

$$\frac{\partial \gamma_{1im}}{\partial t} = \dot{\gamma}_{1im} \tag{164}$$

$$\dot{\gamma}_{1im} = [-e^{-\rho t}(1-\beta)\rho c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta} + e^{-\rho t}(1-\beta)(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta}\beta c^{\beta-1}\dot{c}$$
$$-e^{-\rho t}(1-\beta)\sigma c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma-1}c_{im}^{-\beta}(c_{im}^{1-\beta}\beta c^{\beta-1}\dot{c} + c^{\beta}c_{im}^{1-\beta-1}(1-\beta)\dot{c}_{im})$$
$$+e^{-\rho t}(1-\beta)c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta-1}(-\beta)\dot{c}_{im}](-\frac{p}{p^*})$$

$$\dot{\gamma}_{1im} = -\frac{1}{p^*}e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}[-\rho + \hat{c}(\beta-\sigma\beta) - \hat{c}_{im}(\beta-\sigma\beta+\sigma)] \tag{165}$$

$$\dot{\gamma}_{1im} = \gamma_{1im}[-\rho + \hat{c}(\beta-\sigma\beta) - \hat{c}_{im}(\beta-\sigma\beta+\sigma)] \tag{166}$$

Solving equation (165) it results:

$\partial H/\partial k \overset{!}{=} -\dot{\gamma}_1$

$$\gamma_1 A k^{\alpha-1}\alpha(uh)^{1-\alpha} \overset{!}{=} -\dot{\gamma}_1 \tag{167}$$

dividing by γ_1 and replacing $\widehat{\gamma_1}$ by (162)

$$A\alpha k^{\alpha-1}(uh)^{1-\alpha} = \rho - \hat{c}(\beta - 1 - \sigma\beta) - \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)$$

$$\boxed{\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha\left(\frac{k}{h}\right)^{\alpha-1}u^{1-\alpha} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right)} \tag{168}$$

The domestic growth rate of consumption arises. The derivation of the Keynes Ramsey Rule of the import goods is as follows: Starting with condition (156) and implementing $\dot{\gamma}_{1im}$ by (166) in equation (169) the growth rate is:

$\partial H/\partial k \overset{!}{=} -\dot{\gamma}_{1im}$

$$\gamma_{1im}A(hu)^{1-\alpha}\alpha k^{\alpha-1} \overset{!}{=} -\dot{\gamma}_{1im} \tag{169}$$

$$\boxed{\hat{c}_{im} = \frac{1}{\beta-\sigma\beta+\sigma}\left(A\alpha\left(\frac{k}{uh}\right)^{\alpha-1} - \rho + \hat{c}(\beta-\sigma\beta)\right)} \tag{170}$$

The optimal distribution of human capital between the consumption good sector and the education sector results by equation (157).

$\partial H/\partial u \overset{!}{=} 0$

$$\gamma_1 A k^{\alpha}h^{1-\alpha}u^{-\alpha}(1-\alpha) = \gamma_2 B\bar{B}h \tag{171}$$

After rear ranging to $\boldsymbol{\gamma_2}$ the growth rate of the shadow price is formed as:

$$\gamma_2 = \gamma_1 \frac{Ak^{\alpha} h^{1-\alpha} u^{-\alpha} (1-\alpha)}{B\overline{B}h} \tag{172}$$

$$\boldsymbol{\widehat{\gamma_2} = \widehat{\gamma_1} + \alpha\hat{k} - \alpha\hat{h} - \alpha\hat{u}} \tag{173}$$

The ratio of the shadow priced is:

$$\frac{\gamma_1}{\gamma_2} = \frac{B\overline{B}h}{Ak^{\alpha} h^{1-\alpha} u^{-\alpha} (1-\alpha)} = \frac{B\overline{B}h^{\alpha} u^{\alpha}}{Ak^{\alpha}(1-\alpha)} \tag{174}$$

The following equilibrium condition is deduced by equation (158)

$\partial H/\partial h \overset{!}{=} -\dot{\gamma}_2$

$$\boldsymbol{\gamma_1 Ak^{\alpha} u^{1-\alpha} (1-\alpha) h^{-\alpha} + \gamma_2 B\overline{B}(1-u) \overset{!}{=} -\dot{\gamma}_2} \tag{175}$$

$$\boldsymbol{\frac{\gamma_1}{\gamma_2} Ak^{\alpha} u^{1-\alpha} (1-\alpha) h^{-\alpha} + B\overline{B}(1-u) = -\hat{\gamma}_2} \tag{176}$$

After implementing the ratio $\frac{\gamma_1}{\gamma_2}$ by (231) and $\hat{\boldsymbol{\gamma}}_2$ by equation (173) it arises:

$$\frac{B\overline{B}h}{Ak^{\alpha} h^{1-\alpha} u^{-\alpha} (1-\alpha)} Ak^{\alpha} u^{1-\alpha} (1-\alpha) h^{-\alpha} + B\overline{B}(1-u) = -\hat{\gamma}_1 - \alpha\hat{k} + \alpha\hat{h} + \alpha\hat{u}$$

$$\boxed{\boldsymbol{\hat{u} = B\overline{B}\left(\frac{1-\alpha}{\alpha}\right) + B\overline{B}u - \chi}} \tag{177}$$

Assuming the country is in the foreign trade equilibrium, then the optimal consumption of import goods is equal to the optimal consumption of domestic produced goods, $\boldsymbol{\hat{c} = \hat{c}_{im}}$. Therefore (168)is equal to (170) a general balanced growth path of an open economy arises.

$$\hat{c} = \hat{c}_{im} \tag{178}$$

$$\begin{aligned} \boldsymbol{\frac{1}{(1-\beta+\sigma\beta)}\left[A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right]} &= \\ \boldsymbol{\frac{1}{\sigma(1-\beta)+\beta}\left[A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho + \hat{c}(\beta-\sigma\beta)\right]} \end{aligned} \tag{179}$$

implementing $\boldsymbol{\hat{c}_{im}}$ and resuluted according to $\boldsymbol{\hat{c}}$ leads to:

$$\begin{aligned} \frac{1}{(1-\beta+\sigma\beta)}\left[A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho + \frac{1-\beta+\sigma\beta-\sigma}{\sigma-\sigma\beta+\beta}\left[A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho + \hat{c}(\beta-\sigma\beta)\right]\right] &= \\ \frac{1}{\sigma-\sigma\beta+\beta}\left[A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho + \hat{c}(\beta-\sigma\beta)\right] \end{aligned}$$

$$\boldsymbol{\hat{c} = \frac{1}{\sigma}\left(A\alpha\left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} - \rho\right)} \quad (180)$$

In the equilibrium holds $\boldsymbol{\hat{u} = 0}$.

$$\frac{1-\alpha}{\alpha}B\overline{B} + B\overline{B}u - \chi = 0 \quad (181)$$

$$\boldsymbol{\chi = \frac{1-\alpha}{\alpha}B\overline{B} + B\overline{B}u} \quad (182)$$

Furthermore, in the steady state $\boldsymbol{\hat{c} = \hat{k} = \hat{h}}$ is obtained.

$$\hat{k} = \hat{h} \quad (183)$$

$$Ak^{\alpha-1}(uh)^{1-\alpha} - \chi = (1-u)B\overline{B} \quad (184)$$

$$Ak^{\alpha-1}(uh)^{1-\alpha} - \frac{1-\alpha}{\alpha}B\overline{B} - B\overline{B}u = (1-u)B\overline{B} \quad (185)$$

Shortened by $\boldsymbol{z^* = Ak^{\alpha-1}(uh)^{1-\alpha}}$ it yields:

$$z + \frac{1}{\alpha}B\overline{B} + B\overline{B} - B\overline{B}u = B\overline{B} - B\overline{B}u \quad (186)$$

$$\boldsymbol{z = \frac{1}{\alpha}B\overline{B}} \quad (187)$$

Moreover, in the equilibrium the following relationship is valid.

$$\hat{c} = \hat{k} \quad (188)$$

$$\frac{1}{\sigma}(A\alpha k^{\alpha-1}(hu)^{1-\alpha} - \rho) = Ak^{\alpha-1}(uh)^{1-\alpha} - \chi \quad (189)$$

Substituting $\boldsymbol{A\alpha k^{\alpha-1}(hu)^{1-\alpha}}$ by $\boldsymbol{\alpha z}$ and $\boldsymbol{Ak^{\alpha-1}(uh)^{1-\alpha}}$ by $\boldsymbol{z}$ it results:

$$\frac{1}{\sigma}(\alpha z - \rho) = z - \chi \quad (190)$$

Now $\boldsymbol{z}$ by equation (187) is implemented.

$$\left(B\overline{B} - \rho\right)\frac{1}{\sigma} = \frac{B\overline{B}}{\alpha} - \chi \quad (191)$$

$$\boldsymbol{\chi^* = \frac{B\overline{B}}{\alpha} - \frac{B\overline{B} - \rho}{\sigma}} \quad (192)$$

The optimal allocation of physical capital $\boldsymbol{u}$ results from equalising equation (182) and equation (192).

$$B\overline{B}\frac{1-\alpha}{\alpha}+B\overline{B}u=\frac{B\overline{B}\alpha}{\alpha}-\frac{B\overline{B}-\rho}{\sigma} \quad (193)$$

$$\boxed{\boldsymbol{u}^*=\boldsymbol{1}-\frac{\boldsymbol{1}}{\boldsymbol{\sigma}}\left(\boldsymbol{1}-\frac{\boldsymbol{\rho}}{\boldsymbol{B\overline{B}}}\right)} \quad (\mathbf{194})$$

The balanced growth path of a relatively less developed but open economy by considering (187) is:

$$\boxed{\hat{c}^*=\frac{1}{\sigma}\left(\frac{1}{\alpha}B\overline{B}-\rho\right)} \quad (195)$$

A.3. Open relatively more developed country

The laws of motions for the open relatively more developed economy are:

$$\dot{k}(t)=A(v(t)k(t))^{\alpha}(u(t)h(t))^{1-\alpha}-c(t)-c_{ex}(t)+p^*c_{im}(t) \quad (196)$$

$$\dot{h}(t)=B\overline{B}((1-v(t))k(t))^{\eta}\left((1-u(t))h(t)\right)^{1-\eta} \quad (197)$$

The growth rate for physical capital is as follows by neglecting the time $\boldsymbol{t}$ again.

$$\hat{k}=Av^{\alpha}k^{\alpha-1}(uh)^{1-\alpha}-\frac{c}{k}-\frac{c_{ex}}{k}+p^*\frac{c_{im}}{k}$$

with $\boldsymbol{\chi}=\frac{\boldsymbol{c}}{\boldsymbol{k}}$, $\boldsymbol{\chi_{ex}}=\frac{\boldsymbol{c_{ex}}}{\boldsymbol{k}}$ and $\boldsymbol{\chi_{im}}=\frac{\boldsymbol{c_{im}}}{\boldsymbol{k}}$ it yields

$$\hat{k}=Av^{\alpha}u^{1-\alpha}\left(\frac{k}{h}\right)^{\alpha-1}-\chi-\chi_{ex}+p^*\chi_{im} \quad (198)$$

Implementing $\boldsymbol{x_1}=\frac{\boldsymbol{vk}}{\boldsymbol{uh}}$ the growth rate can be written shortened as:

$$\boxed{\hat{k}=Ax_1^{\alpha}\frac{uh}{k}-\chi-\chi_{ex}+p^*\chi_{im}} \quad (199)$$

The human capital grow as follows:

$$\hat{h}=B\overline{B}\left[(1-v)\frac{k}{h}\right]^{\eta}(1-u)^{1-\eta} \quad (200)$$

A substitution of $\boldsymbol{x_2} = \frac{(\mathbf{1-v})\boldsymbol{k}}{(\mathbf{1-u})\boldsymbol{h}}$ leads to:

$$\boxed{\hat{h} = B\overline{B}x_2^{\eta}(1-u)} \tag{201}$$

The Hamiltonian in the relatively more developed country solves the optimization problem.

$$\begin{aligned}\mathbb{H} = \quad & \boldsymbol{e^{-\rho t}\frac{(c^{\beta}c_{im}^{1-\beta})^{1-\sigma}}{1-\sigma}} \\ & \boldsymbol{+\gamma_1(A(vk)^{\alpha}(uh)^{1-\alpha} - c - c_{ex} + p^{*}c_{im})} \\ & \boldsymbol{+\gamma_2 B\overline{B}[(1-v)k]^{\eta}[(1-u)h]^{1-\eta}}\end{aligned} \tag{202}$$

Even here the well-known first order conditions are:

$$\frac{\partial\mathbb{H}}{\partial c} \overset{!}{=} 0 \tag{203}$$

$$\frac{\partial\mathbb{H}}{\partial c_{im}} \overset{!}{=} 0 \tag{204}$$

$$\frac{\partial\mathbb{H}}{\partial v} \overset{!}{=} 0 \tag{205}$$

$$\frac{\partial\mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma}_1 \tag{206}$$

$$\frac{\partial\mathbb{H}}{\partial k} \overset{!}{=} -\dot{\gamma}_{1im} \tag{207}$$

$$\frac{\partial\mathbb{H}}{\partial u} \overset{!}{=} 0 \tag{208}$$

$$\boldsymbol{\frac{\partial\mathbb{H}}{\partial h} \overset{!}{=} -\dot{\gamma}_2} \tag{209}$$

Starting with equation (203) we get:

$\partial H/\partial c \overset{!}{=} 0$

$$\gamma_1 = e^{-\rho t}\beta c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \tag{210}$$

To get the Keynes-Ramsey-Rule the shadow price $\boldsymbol{\gamma_1^{*}}$ by equation (210) needs to be differentiated with respect to time $\boldsymbol{t}$.

$$\frac{\partial\gamma_1}{\partial t} = \dot{\gamma}_1 \tag{211}$$

$$\begin{aligned}\dot{\gamma}_1 = \; & -e^{-\rho t}\beta\rho c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} + e^{-\rho t}\beta(\beta-1)c^{\beta-2}\dot{c}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \\ & + e^{-\rho t}\beta c^{\beta-1}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}(1-\beta)c_{im}^{1-\beta-1}\dot{c}_{im} \\ & - e^{-\rho t}\beta\sigma c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-1-\sigma}(c_{im}^{1-\beta}\beta c^{\beta-1}\dot{c} + c^{\beta}c_{im}^{1-\beta-1}(1-\beta)\dot{c}_{im})\end{aligned}$$

$$\dot{\gamma}_1 = e^{-\rho t}\beta c^{\beta-1}c_{im}^{1-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}[-\rho + \hat{c}(\beta - 1 - \sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)] \quad (212)$$

rearranged follows:

$$\dot{\gamma_1} = \gamma_1[-\rho + \hat{c}(\beta - 1 - \sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)]$$

It results the growth rate of the shadow price of good one.

$$\widehat{\gamma_1} = -\rho + \hat{c}(\beta - 1 - \sigma\beta) + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) \quad (213)$$

The consumption of the imported goods influences the life time utility as follows, see condition (204).

$\partial H/\partial c_{im} \stackrel{!}{=} 0$

$$p^*\gamma_1 \mathrel{\widehat{=}} \gamma_{1im} = -e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma} \quad (214)$$

Differentiating equation (214) with respect to time it results:

$$\frac{\partial\gamma_{1im}}{\partial t} = \dot{\gamma}_{1im} \quad (215)$$

$$\begin{aligned}\dot{\gamma}_{1im} = \; & [-e^{-\rho t}(1-\beta)\rho c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta} + e^{-\rho t}(1-\beta)(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta}\beta c^{\beta-1}\dot{c} \\ & - e^{-\rho t}(1-\beta)\sigma c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma-1}c_{im}^{-\beta}(c_{im}^{1-\beta}\beta c^{\beta-1}\dot{c} + c^{\beta}c_{im}^{1-\beta-1}(1-\beta)\dot{c}_{im}) \\ & + e^{-\rho t}(1-\beta)c^{\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}c_{im}^{-\beta-1}(-\beta)\dot{c}_{im}](-\frac{p}{p^*})\end{aligned}$$

$$\dot{\gamma}_{1im} = -\frac{1}{p^*}e^{-\rho t}(1-\beta)c^{\beta}c_{im}^{-\beta}(c^{\beta}c_{im}^{1-\beta})^{-\sigma}[-\rho + \hat{c}(\beta - \sigma\beta) - \hat{c}_{im}(\beta - \sigma\beta + \sigma)] \quad (216)$$

$$\dot{\gamma}_{1im} = \gamma_{1im}[-\rho + \hat{c}(\beta - \sigma\beta) - \hat{c}_{im}(\beta - \sigma\beta + \sigma)] \quad (217)$$

The condition (205) leads to the optimal allocation of physical capital v^*.

$\partial H/\partial v \stackrel{!}{=} 0$

$$\gamma_1 A\alpha v^{\alpha-1}k^{\alpha}(uh)^{1-\alpha} = \gamma_2 B\bar{B}\eta(1-v)^{\eta-1}k^{\eta}[(1-u)h]^{1-\eta} \quad (218)$$

It results the ratio of both shadow prices.

$$\frac{\gamma_2}{\gamma_1} = \frac{A\alpha v^{\alpha-1} k^{\alpha} (uh)^{1-\alpha}}{B\overline{B}\eta(1-v)^{\eta-1} k^{\eta} [(1-u)h]^{1-\eta}} \tag{219}$$

$$\boldsymbol{\frac{\gamma_2}{\gamma_1} = \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\overline{B}\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} = \frac{A\alpha x_1^{\alpha-1}}{B\overline{B}\eta x_2^{\eta-1}}} \tag{220}$$

$$\boldsymbol{\gamma_2 = \gamma_1 \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\overline{B}\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} \Leftrightarrow \gamma_1 = \gamma_2 \frac{B\overline{B}\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}}{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}} \tag{221}$$

The growth rate of the shadow price of good one is deduced from equation (206).

$\partial H/\partial k \overset{!}{=} -\dot{\gamma}_1$

$$\boldsymbol{\gamma_1 A v^{\alpha} k^{\alpha-1} \alpha (uh)^{1-\alpha} + \gamma_2 B\overline{B}(1-v)^{\eta} k^{\eta-1} \eta [(1-u)h]^{1-\eta} \overset{!}{=} -\dot{\gamma}_1} \tag{222}$$

$$\boldsymbol{A\alpha v^{\alpha} k^{\alpha-1} (uh)^{1-\alpha} + \frac{\gamma_2}{\gamma_1} B\overline{B}\eta(1-v)^{\eta} k^{\eta-1} [(1-u)h]^{1-\eta} = -\hat{\gamma}_1}$$

The implementation of the shadow price ratio $\boldsymbol{\gamma_2/\gamma_1}$ by equation (220) leads to:

$$A\alpha v^{\alpha} u^{1-\alpha} \left(\frac{k}{h}\right)^{\alpha-1} + \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\overline{B}\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} B\overline{B}\eta(1-v)^{\eta} k^{\eta-1} [(1-u)h]^{1-\eta} = -\hat{\gamma}_1$$

$$\boldsymbol{\hat{\gamma}_1 = -A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1} \Leftrightarrow \hat{\gamma}_1 = -A\alpha x_1^{\alpha-1}} \tag{223}$$

Combining equation (222) with $\boldsymbol{\gamma_2}$ by (221) and $\boldsymbol{\dot{\gamma}_1}$ by (212) the Keynes-Ramsey-Rule follows:

$$\gamma_1 A\alpha v^{\alpha} \left(\frac{k}{h}\right)^{\alpha-1} u^{1-\alpha} + \gamma_1 \frac{A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1}}{B\overline{B}\eta\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} B\overline{B}\eta(1-v)^{\eta} k^{\eta-1} [h(1-u)]^{1-\eta}$$
$$\overset{!}{=} -\gamma_1[-\rho + \hat{c}(\beta - 1 - \sigma\beta) + \hat{c}_{im}(1 - \beta + \sigma\beta - \sigma)] \tag{224}$$

$$\boxed{\boldsymbol{\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha\left(\frac{vk}{uh}\right)^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right)}} \tag{225}$$

The domestic consumption depends even in the more developed country on the import goods. To calculate the growth rate of the import goods the first order condition (207) is needed.

$\partial H/\partial k \overset{!}{=} -\dot{\gamma}_{1im}$

$$\boldsymbol{\gamma_{1im} A(hu)^{1-\alpha} \alpha v^{\alpha} k^{\alpha-1} + \gamma_2 B\overline{B}[h(1-u)]^{1-\eta} \eta (1-v)^{\eta} k^{\eta-1} \overset{!}{=} -\dot{\gamma}_{1im}} \quad (226)$$

Implementing $\boldsymbol{\dot{\gamma}_{1im}}$ and $\boldsymbol{\gamma_2}$ by (217) and (221) in (226) it yields:

$$\gamma_{1im} A(hu)^{1-\alpha} \alpha v^{\alpha} k^{\alpha-1} + \gamma_{1im} \frac{A\alpha \left(\frac{vk}{uh}\right)^{\alpha-1}}{B\overline{B}\eta \left(\frac{(1-v)k}{(1-u)h}\right)^{\eta-1}} B\overline{B}[h(1-u)]^{1-\eta} \eta (1-v)^{\eta} k^{\eta-1}$$
$$= -\dot{\gamma}_{1m}$$

$$\boxed{\boldsymbol{\hat{c}_{im} = \frac{1}{\beta - \sigma\beta + \sigma} \left(A\alpha \left(\frac{vk}{uh}\right)^{\alpha-1} - \rho + \hat{c}(\beta - \sigma\beta) \right)}} \quad (227)$$

It follows the optimal distribution of human capital between the consumption and the education sector by equation (208).

$\partial H/\partial u \overset{!}{=} 0$

$$\boldsymbol{\gamma_1 A(1-\alpha)(vk)^{\alpha} h^{1-\alpha} u^{-\alpha} = \gamma_2 B\overline{B}(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta} h^{1-\eta}} \quad (228)$$

$$\boldsymbol{\frac{\gamma_2}{\gamma_1} = \frac{A(1-\alpha)(vk)^{\alpha} h^{1-\alpha} u^{-\alpha}}{B\overline{B}(1-\eta)[(1-v)k]^{\eta}(1-u)^{-\eta} h^{1-\eta}}} \quad (229)$$

The ratio of the shadow prices is:

$$\frac{\gamma_2}{\gamma_1} = \frac{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}}{B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}} = \frac{A(1-\alpha)x_1^{\alpha}}{B\overline{B}(1-\eta)x_2^{\eta}} \quad (230)$$

$$\boldsymbol{\gamma_1 = \gamma_2 \frac{B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}}{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}} \Leftrightarrow \gamma_2 = \gamma_1 \frac{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}}{B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}} = \gamma_1 \frac{A(1-\alpha)x_1^{\alpha}}{B\overline{B}(1-\eta)x_2^{\eta}}}$$
$$(231)$$

The growth rate of the shadow price of good 2 follows.

$$\hat{\gamma}_2 = \hat{\gamma}_1 + \alpha\hat{x}_1 - \eta\hat{x}_2 \quad (232)$$

After that the ratios by (220) and (230) get equalized.

$$\frac{A\alpha x_1^{\alpha-1}}{B\overline{B}\eta x_2^{\eta-1}} = \frac{A(1-\alpha)x_1^{\alpha}}{B\overline{B}(1-\eta)x_2^{\eta}} \tag{233}$$

$$\boxed{\boldsymbol{\frac{1-\alpha}{\alpha}x_1 = \frac{1-\eta}{\eta}x_2}} \tag{234}$$

The last first order condition by (209) is:

$\partial H/\partial h \overset{!}{=} -\dot{\gamma}_2$

$$\boldsymbol{\gamma_1 A(1-\alpha)(vk)^{\alpha}u^{1-\alpha}h^{-\alpha} + \gamma_2 B\overline{B}(1-\eta)[(1-v)k]^{\eta}(1-u)^{1-\eta}h^{-\eta} \overset{!}{=} -\dot{\gamma}_2} \quad (\mathbf{235})$$

Substituting $\boldsymbol{\gamma_1}$ by (231) it yields:

$$\gamma_2 \frac{B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}}{A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha}} A(1-\alpha)\left(\frac{vk}{uh}\right)^{\alpha} u + \gamma_2 B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta}(1-u) = -\dot{\gamma}_2$$

$$\boldsymbol{\hat{\gamma}_2 = -B\overline{B}(1-\eta)\left(\frac{(1-v)k}{(1-u)h}\right)^{\eta} \Leftrightarrow \hat{\gamma}_2 = -B\overline{B}(1-\eta)x_2^{\eta}} \tag{236}$$

Then equation (232) is implemented in equation (236).

$$\hat{\gamma}_1 + \alpha\hat{x}_1 - \eta\hat{x}_2 = -B\overline{B}(1-\eta)x_2^{\eta} \tag{237}$$

Inserting equation (213) in this equation it yields:

$$\boxed{B\overline{B}(1-\eta)x_2^{\eta} = \rho - \hat{c}(\beta - 1 - \sigma\beta) - \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) - \alpha\hat{x}_1 + \eta\hat{x}_2} \tag{238}$$

The following system of equations describes the equilibrium of a relatively more developed open economy.

$$\hat{k} = Ax_1^{\alpha}\frac{uh}{k} - \chi - \chi_{ex} + p^{*}\chi_{im} \tag{239}$$

$$\hat{h} = B\overline{B}x_2^{\eta}(1-u) \tag{240}$$

$$x_1(1-\alpha)/\alpha = x_2(1-\eta)/\eta \tag{241}$$

$$\hat{c} = \frac{1}{(1-\beta+\sigma\beta)}\left(A\alpha x_1^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right) \tag{242}$$

$$B\overline{B}(1-\eta)x_2^{\eta} = \rho - \hat{c}(\beta - 1 - \sigma\beta) - \hat{c}_{im}(1-\beta+\sigma\beta-\sigma) - \alpha\hat{x}_1 + \eta\hat{x}_2 \tag{243}$$

Assuming a foreign trade equilibrium, $\hat{\boldsymbol{c}} = \hat{\boldsymbol{c}}_{\boldsymbol{im}}$ holds. By using equation (225) and (227) the general balanced growth path arises.

$$\hat{c} = \hat{c}_{im}$$

$$\frac{1}{(1-\beta+\sigma\beta)}\left[A\alpha x_1^{\alpha-1} - \rho + \hat{c}_{im}(1-\beta+\sigma\beta-\sigma)\right] =$$

$$= \frac{1}{\sigma(1-\beta)+\beta}\left[A\alpha x_1^{\alpha-1} - \rho + \hat{c}(\beta-\sigma\beta)\right] \tag{244}$$

$$\boxed{\hat{\boldsymbol{c}} = \frac{\boldsymbol{1}}{\boldsymbol{\sigma}}\left(\boldsymbol{A\alpha x_1^{\alpha-1}} - \boldsymbol{\rho}\right)} \tag{245}$$

Equation (241) shows the reactions of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in the long run. The growth rate of $\boldsymbol{x_1}$ is formed and it holds that:

$$\hat{x}_1 = \hat{x}_2 = 0 \tag{246}$$

In the steady state holds: $\hat{\boldsymbol{c}} = \widehat{\boldsymbol{k}} = \widehat{\boldsymbol{h}}$. From condition (243) follows $\boldsymbol{x_1^*}$ and is calculated with $\hat{\boldsymbol{c}} = \hat{\boldsymbol{c}}_{\boldsymbol{im}}$.

$$B\left(1+\overline{B}\right)(1-\eta)x_2^{\eta} = \rho - \hat{c}(\beta - 1 - \sigma\beta) - \hat{c}(1-\beta+\sigma\beta-\sigma) \tag{247}$$

$$\boldsymbol{x_2^{\eta}} = \frac{\boldsymbol{1}}{\boldsymbol{B}(\boldsymbol{1}+\overline{\boldsymbol{B}})(\boldsymbol{1}-\boldsymbol{\eta})}\left[\boldsymbol{\rho} + \hat{c}(-\boldsymbol{\beta} + \boldsymbol{1} + \boldsymbol{\sigma\beta} + \boldsymbol{\beta} - \boldsymbol{\sigma\beta} + \boldsymbol{\sigma})\right]$$

$$\boldsymbol{x_2^*} = \left(\frac{\boldsymbol{\rho} + \boldsymbol{\sigma}\hat{\boldsymbol{c}}}{\boldsymbol{B}\left(\boldsymbol{1}+\overline{\boldsymbol{B}}\right)(\boldsymbol{1}-\boldsymbol{\eta})}\right)^{\frac{\boldsymbol{1}}{\boldsymbol{\eta}}} \tag{248}$$

Resulting from equation (241) $\boldsymbol{x_1^*}$ follows, by implementing $\boldsymbol{x_2^*}$.

$$x_1^* = \frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\sigma\hat{c}}{B\left(1+\overline{B}\right)(1-\eta)}\right)^{\frac{1}{\eta}} \tag{249}$$

Equation (245) shows the balanced growth path and it yields:

$$\hat{c} = \frac{1}{\sigma}\left(A\alpha\left[\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\sigma\hat{c}}{B\left(1+\overline{B}\right)(1-\eta)}\right)^{\frac{1}{\eta}}\right]^{\alpha-1} - \rho\right) \tag{250}$$

$$\boxed{\hat{\boldsymbol{c}}^* = \frac{\boldsymbol{1}}{\boldsymbol{\sigma}}\left(\left[\boldsymbol{A^{\eta}\alpha^{\alpha\eta}}(\boldsymbol{1}-\boldsymbol{\eta})^{(\boldsymbol{1}-\boldsymbol{\eta})(\boldsymbol{1}-\boldsymbol{\alpha})}(\boldsymbol{\eta}(\boldsymbol{1}-\boldsymbol{\alpha}))^{\boldsymbol{\eta}(\boldsymbol{1}-\boldsymbol{\alpha})}(\boldsymbol{B}(\boldsymbol{1}+\overline{\boldsymbol{B}}))^{\boldsymbol{1}-\boldsymbol{\alpha}}\right]^{\frac{\boldsymbol{1}}{\boldsymbol{1}+\boldsymbol{\eta}-\boldsymbol{\alpha}}} - \boldsymbol{\rho}\right)} \tag{251}$$

Placeholder $\boldsymbol{M} = \left[\boldsymbol{A^{\eta}\alpha^{\alpha\eta}}(\boldsymbol{1}-\boldsymbol{\eta})^{(\boldsymbol{1}-\boldsymbol{\eta})(\boldsymbol{1}-\boldsymbol{\alpha})}(\boldsymbol{\eta}(\boldsymbol{1}-\boldsymbol{\alpha}))^{\boldsymbol{\eta}(\boldsymbol{1}-\boldsymbol{\alpha})}(\boldsymbol{B}(\boldsymbol{1}+\overline{\boldsymbol{B}}))^{\boldsymbol{1}-\boldsymbol{\alpha}}\right]^{\frac{\boldsymbol{1}}{\boldsymbol{1}+\boldsymbol{\eta}-\boldsymbol{\alpha}}}$ clarifies the following equations. The balanced allocation of human capital $\boldsymbol{u}$ is derivated by the condition $\hat{\boldsymbol{c}} = \widehat{\boldsymbol{h}}$ by equation (240) considering $\boldsymbol{x_2^*}$ and $\hat{\boldsymbol{c}}^*$.

$$\frac{1}{\sigma}(M-\rho)=B(1+\overline{B})\left(\left(\frac{\rho+\sigma\frac{1}{\sigma}(M-\rho)}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\right)^{\alpha}(1-u) \tag{253}$$

$$\boxed{\boldsymbol{u^{*}=\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}}} \tag{253}$$

Additionally, the equilibrium leads to the same growth rates of consumption and physical capital $\hat{\boldsymbol{c}}=\widehat{\boldsymbol{k}}$. By this condition the optimal ratio of physical capital and human capital is calculated, by using $\boldsymbol{x_1^*}$, $\boldsymbol{x_2^*}$ and $\hat{\boldsymbol{c}}^*$ in equation (239).

$$\boxed{\chi^{*}=\frac{1}{\sigma}\left(\frac{A\alpha\sigma[-\eta\rho+M(\eta+\sigma-1)+\rho]\left(\frac{\alpha(\eta-1)\left(\frac{M}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}}{(\alpha-1)\eta}\right)^{\alpha-1}}{\rho(\alpha-\eta)+M(\alpha(\sigma-1)+\eta)}-M+\rho\right)} \tag{254}$$

By implementing $\boldsymbol{x_1^*}$, $\boldsymbol{x_2^*}$ and with $\boldsymbol{\frac{k}{h}=ux_1+(1-u)x_2}$ in $\boldsymbol{v=\frac{vk}{uh}\frac{uh}{k}}$ it yields:

$$v=\frac{\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\frac{M\sigma-(1-\eta)(M-\rho)}{\sigma M}}{\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}\frac{\alpha(1-\eta)}{\eta(1-\alpha)}\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}+\left(1-\frac{\sigma M-(1-\eta)(M-\rho)}{\sigma M}\right)\left(\frac{\rho+\frac{1}{\sigma}\sigma(M-\rho)}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}} \tag{255}$$

It results the optimal allocation of physical capital $\boldsymbol{v}^*$.

$$\boxed{v^{*}=\frac{\alpha(1-\eta)\left(\frac{M}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(M\sigma-(1-\eta)(M-\rho)\right)}{(1-\alpha)\eta M\sigma\left(\frac{\alpha(1-\eta)\left(\frac{M}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(M\sigma-(1-\eta)(M-\rho)\right)}{(1-\alpha)\eta M\sigma}+\left(\frac{M}{B(1+\overline{B})(1-\eta)}\right)^{\frac{1}{\eta}}\left(1-\frac{M\sigma-(1-\eta)(M-\rho)}{M\sigma}\right)\right)}} \tag{256}$$