Determining the Efficiency Frontier

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Abstract. *This article tries to answer an actual issue inducted by the title portfolio management. How do we combine the risky assets with the ones without risk, which are the portfolios selection criteria, which are their performances and the choices of the rational investor.*

Key words: efficiency frontiers; efficient portfolio; assets without risk.

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JEL Classification: G11, G14.

Portfolio management allows the achievement of a maximum profitability per risk unit. The manager of a portfolio follows the identification of the efficient portfolios starting from the premise that there is an efficient portfolio at a certain profitability which is preferred among all the other portfolios which have an equal profitability with this one. Starting from this content the efficiency frontier offers us a relation between the profitability and the risk of the dominant portfolios. It allows us to know which are the gain expectancies of the investments for a certain level of assumed risk.

There are two big identifiable categories of efficiency frontiers:

1. The first category is the efficiency frontier built only on risky assets, starting from the premise that the investor makes placements only in risky assets preferring the risk;
2. The second category is the efficiency frontier calculated for the risky assets and the assets without risk, when the investor makes its placements in assets without risk in order to diminish the assumed risk.

The theories that lays at the foundation of determining the efficient portfolios consisting of risky assets are those of Markowitz and Sharpe.

Markowitz considers that the selection process of the portfolio can be divided in two stages, specifically: the analysis of the value titles and the establishment of some scenarios regarding the profitability evolution and that of the future risk, the process of optimum portfolio selection based on the previous predictions. He does not accept the rule of actualized value maximization of the future benefits because this does not take into account the risk and the investors' attitude towards risk. A criterion is proposed aiming to increase simultaneously the specific profitability of the mobile values and to diminish the risk associated with it. Markowitz suggests that the process of portfolio selection should be approached in connection to previous estimations of the titles future performances. The analysis of these estimations in order to determine a group of efficient portfolios and the selection from this multitude of efficient portfolios that meet the investors' preferences represents the meaning of Markowitz's theory.

This model tries to solve the problem of portfolio management which consists of determining the efficient portfolios. A portfolio is efficient if no other ensures a better profitability for the same risk or the same profitability at a lower risk.

The portfolio analysis requires a large quantity of information and Sharpe was trying to diminish the number of these information using a set of simplifying hypothesis. The diagonal model answers these requirements starting from a simple presentation of the correlation that exists among the profitability evolution and the placement mobile value risk and a macroeconomic factor. Thus, we eliminate the big number of information necessary for grouping the inter-correlations between the titles, taken two by two. The profitability of a title is in a linear relation with a macroeconomic factor and the associated risk can be structured in specific risk and systematic risk. The specific risk can be taken away through diversification and this is where the investment talent of the portfolio manager manifests and the systematic risk is a characteristic of the economic environment and can not be eliminated through diversification. In Sharpe's model, in contrast with Markowitz's model, the negative balances are admitted too, representing the loan to the interest installment without risk to obtain the funds necessary for the risky assets with big profitability. This model introduces as a coefficient to measure the correlation among the profitability and the risk of the title and the considered macroeconomic factor, the volatility indicator. The introduction in the portfolio of the asset without risk according to CAPM model leads to the appearance of a new efficiency frontier.

A. The efficiency frontier with actives without risk

The portfolio management is uni-periodical which allows the determination of an optimal structure for the period taken into account. The calculations of profitability expectance and that of variance-covariance matrix correspond to this period.

The profitability of the assets without risk is the one that corresponds to the portfolio structure study period. If the portfolio is restructured at three months, for example, the profitability of the assets without risk can be the interest installment for the thesaurus bills issued at three months. The interest installment is considered unique for all the investors categories and also considers, theoretically, that the interest installment for deposits is equal with the interest installment for loans.

Be the profitability vector N actives:

$$R = \begin{bmatrix} E(R_1) \\ \vdots \\ E(R_n) \end{bmatrix}$$

The vector of the investment proportion in risky assets is:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Variance-covariance matrix:

$$\Omega = \begin{bmatrix} \sigma_{1i}^2 \ldots\ldots\ldots\ldots \sigma_{1N} \\ \vdots \\ \sigma_{1j} \ldots \sigma_{1j}^2 \ldots \sigma_{iN} \\ \vdots \\ \sigma_{1N} \ldots\ldots\ldots\ldots \sigma_N^2 \end{bmatrix}$$

The profitability expectancy of the portfolio is:

$$E(R_p) = \overline{R}^T \times x = x^T \times \overline{R}$$

where:

x^T is the transpose of x.

Portfolio dispersion or variance is:

$$V(R_p) = x^T \times \Omega \times X$$

This modality allows the matrix calculation for profitability and risk, using the variance-covariance matrix.

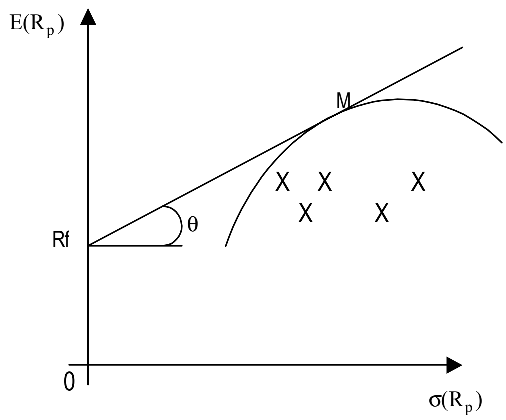
The computation of efficiency frontier with assets without risk

Due to Tobin's studies we know that the efficiency frontier is a line. The problem is to determine the portfolio structure which are situated on this line and to determine the equations of profitability/risk.

a) Graphic solution:

A method to solve the problem consist of:

- identification of efficiency frontier composed by risky assets and risk less assets;
- the determination of the line characteristic for the risky assets which is tangent with the respective efficiency frontier.



His efficiency line (R_f, M) offers a secant of superior profitability to the efficiency curve without asset without risk. The assets without risk ameliorate the relation profitability-risk.

b) A direct mathematical solution:

This consists of direct maximization of the R_f line thus:

$$\text{Max}\theta = \frac{E(\overline{R_p}) - R_f}{\sqrt{X(R_p)}} \quad \text{cu .....} \sum_{i=1}^{n} x_i = 1$$

To introduce the asset without risk profitability in the objective function we consider that $R_f = \sum_{i=1}^{n} x_i \times R_f$

We will have $\theta = \frac{\sum x_i \times R_i - \sum x_i \times R_f}{\sqrt{\sum x_i \times x_j \times \sigma_{ij}}}$

$$\theta = \frac{\sum_{i=1}^{n} x_i (\overline{R_i} - R_f)}{\sqrt{\sum_{i=1} \sum_{j=1} x_i \times x_j \times \sigma_{ij}}}$$

For maximization:

$$\frac{\sum x_i(\overline{R_i}-R_f)}{\sqrt{\sum_{i=1}\sum_{j=1} x_i \times x_j \times \sigma_{ij}}} = x^T(\overline{R}-\overline{R_f})[x^\sigma \times \Omega \times x]^{\frac{1}{2}}$$

The differential of first order is:

$$\frac{d\theta}{dx} = (\overline{R}-\overline{R_f})[x^T \times \Omega \times x]^{-\frac{1}{2}} +$$

$$+(\frac{1}{2})2\Omega \times x[x^T \times \Omega \times x]^{-\frac{3}{2}}(x^T(\overline{R}-\overline{R_f}))$$

where $\overline{R_f}$ is a vector of N dimension

Multiplying the equation with: $[x^T \times \Omega \times x]^{\frac{1}{2}}$ we get:

$$(\overline{R}-\overline{R_f}) - \Omega \times x[x^T \times \Omega \times x]^{-1}(x^T(\overline{R}-\overline{R_f})) = 0$$

The calculation consist in determining the differences between the column vectors of N dimension:

$$(\overline{R}-\overline{R_f}) - [x^T \times \Omega \times x]^{-1}(x^T(\overline{R}-\overline{R_f}))\Omega \times x = 0$$

Noting $\lambda = (x^T(\overline{R}-\overline{R_f})(\overline{R}-\overline{R_f})[x^T \times \Omega \times x]^{-1}$ the prime of risk per risk unit becomes:

$$(\overline{R}-\overline{R_f}) - \lambda \times \Omega \times x = 0$$

A last change of variable leads to:

$$\lambda \times x = z$$

$$R - R_f = \Omega \times z$$

$$\begin{cases} R_1 - R_f = z_1 \times \sigma_{11} + z_2 \times \sigma_{12} + \ldots + z_n \times \sigma_n \\ \vdots \\ \overline{R_N} - R_f = z_1 \times \sigma_{N1} + z_2 \times \sigma_{N1} + \ldots + z_N \times \sigma_{NN} \end{cases}$$

It is a system in z to find out the investments proportion we make the change of variable:

$$x_i = \frac{z_i}{\sum z_i}$$

$$\sum x_i \times \lambda = \sum z_i \Rightarrow \lambda = \sum z_i \text{ dacă} \sum x_{i'} = 1$$

The calculation of the efficiency frontier without risk assets

This frontier is hard to compute, because it is not the equation of a line but of a curve of II grade.

The most used method is the Merton method(1972).

He considers the efficiency frontier can be built starting from the combination of two efficient portfolios consisting of risky assets.

We will follow:

$$\text{Min} \frac{1}{2} x^T \times \Omega \times x$$

With the conditions:

$$x^T \times \overline{R} = E(\overline{R_p})$$

$$X^T \times \bar{1} = 1$$

where:

$\bar{1}$ = a column vector of 1.

$E(\overline{R_p})$ = the expectancy of the fix profitability for the portfolio

It is calculated the langrangian:

$$L = \frac{1}{2} \times X^T \times \Omega \times X + \lambda[E(R_p) - X^T\overline{R}] + \gamma[1 - X^T \times 1]$$

where the condition of first order:

$$\begin{cases} \frac{dL}{dx} = \Omega \times x - \lambda \times \overline{R} - \gamma \times \bar{1} = 0 & (1) \\ \frac{dL}{d\lambda} = E(R_p) - x^T \times \overline{R} = 0 & (2) \\ \frac{dL}{d\gamma} = 1 - \bar{x}^T \times \bar{1} = 0 & (3) \end{cases}$$

From the equation (1) we have:

$$x = \lambda \times \Omega^{-1} \times \bar{r} + \gamma \times \Omega^{-1} \times \bar{1} \quad (4)$$

Multiplying this equation with $\overline{R}^T$:

$$\overline{R}^T X = \lambda(\overline{R}^T \times \Omega^{-1} \times \overline{R}) + \gamma(\overline{R}^R \times \Omega^{-1} \times \bar{1})$$

Replacing $\overline{R}^T x$ in equation (2) we get:

$$E(\overline{R}_p) = \lambda(\overline{R}^T \times \Omega \times \overline{R}) + \gamma(\overline{R}^T \times \Omega^{-1} \times \bar{1}) \quad (5)$$

Multiplying the equation (1) with the linear vector 1:

$$\bar{1}^T x = \lambda(\bar{1}^T \times \Omega^{-1} \times \overline{R}) + \gamma(\bar{1}^T \times \Omega^{-1} \times \bar{1})$$

And using equation (3):

$$1 X = \lambda(\bar{1}^T \times \Omega^{-1} \times \overline{R}) + \gamma(\bar{1}^T \times \Omega^{-1} \times \bar{1}) \quad (6)$$

The equation system:

$$\left.\begin{matrix} E(\overline{R}_p) = \lambda(\overline{R}^T \times \Omega^{-1} \times \overline{R}) + \gamma(\overline{R}^T \times \Omega^{-1} \times \bar{1}) \\ \text{şi} \\ 1 = \lambda(\bar{1}^T \times \Omega^{-1} \times \overline{R}) + \gamma(\bar{1}^T \times \Omega^{-1} \times \bar{1}) \end{matrix}\right\} \Rightarrow$$

can be solved by using the determinants method

$$\begin{vmatrix} \overline{R}^T \Omega^{-1} \overline{R} & \overline{R}^T \Omega^{-1} \bar{1} \\ \bar{1}^T \Omega^{-1} \overline{R} & \bar{1}^T \Omega^{-1} \bar{1} \end{vmatrix} =$$

$$(\overline{R}^T \times \Omega^{-1} \times \overline{R}) \times (\bar{1}^T \times \Omega^{-1} \bar{1}) - (\overline{R}^T \times \Omega^{-1} \times \bar{1})^2 = B \times C - A^2$$

$$\lambda = \frac{\begin{vmatrix} \overline{R}^T \Omega^{-1} \overline{R} & E(\overline{R_p}) \\ \bar{1}^T \Omega^{-1} \overline{R} & 1 \end{vmatrix}}{D} = \frac{B - A \times E(\overline{R_p}) - A}{D}$$

$$\gamma = \frac{\begin{vmatrix} E(\overline{R_p}) & \overline{R}^T \Omega^{-1} \bar{1} \\ 1 & \bar{1}^T \Omega^{-1} \bar{1} \end{vmatrix}}{D} = \frac{B - A \times E(\overline{R_p})}{D}$$

where:

$$A = R^{T} \times \Omega^{-1} \times \bar{1};$$
$$B = R^{T} \times \Omega^{-1} \times R;$$
$$C = \bar{1}^{T} \times \Omega^{-1} \times \bar{1};$$
$$D = B \times C - A^{2}.$$

The proportion inside the efficient portfolio is determined starting from the equation (4) by replacing the values:

$$x = \frac{[C \times E(\bar{R}_p) - A] \times \Omega^{-1} \times \bar{R} + [B - A \times E(\bar{R}_p)] \times \Omega^{-1} \times \bar{1}}{D}$$

$$x = \frac{B \times \Omega^{-1} \times \bar{1} - A \times \Omega^{-1} \times \bar{R}}{D} + \frac{C \times \Omega^{-1} \times \bar{R} - A \times \Omega^{-1} \times \bar{1}}{D} E(\overline{R_p})$$

$$x = g + h \times E(\bar{R}_p)$$

If we consider that $E(R_p) = 0$, then g represents the optimum proportion in a portfolio with a null secant of profitability. For $E(R_p) = 1$ results that $x = g + h$ is the optimum proportion for a portfolio with the secant of profitability equal to 1.

Presenting the concepts represents just the beginning of the portfolio management analysis. The mirage of low risk investments but with high returns will always exist despite our desire to present the objective, mathematic content of the choice. What I want to underline is that there will always be those that will not have a rational behavior and that will want eventually even to break the rules for an immediate gain. The human nature gives up in front of the wish to get rich making curious choices. Would we be able to quantify these things also? This is a challenge.

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