Geometrical Methods in the Study of Some Dynamical Systems with Applications in Economy

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Abstract. *In this paper dynamical patterns of some economical processes are being analysed. Starting from the system of differential equations the stability of the system is being analysed. The study of the stability of some dynamical systems has been a constant interest for the author in the last fifteen years.*

Key words: stability; Leapunov function; dynamic system the phase plan.

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JEL Codes: C02, C62.
REL Codes: 5C, 10A.

1. Introductions

The phenomena of the ecconomical systems are often described with the help of the fields. In the mathematical physics, by field we often mean a function of time t and position x. The field is vectorial if its values are vectors. When the fields don't depend on x, they become functions only of time. Their set is associated to the dynamic that takes place in the dynamic system. The dynamic is described by the position vectors $x_i(t^*)$ and speed $v_i(t*)$, $i=\overline{1,N}$. The set $\{x_i(t*),v_i(t*)\}_{i=\overline{1,N}}$ it is called the state (phase) where it is situated at the moment $t*$.

This way, the dynamic was modelled by a continuous order, in proportion with the time parameter of the states of the system. From the physical point of view, the space of phases is the set of all the possible states of a dynamic system. The functions that compose a state are called functions or (variables) of state. The short and synthetic characterization of the evolution of an ecconomical system can be done with the help of some concepts that model properties corresponding to the dynamic associated system: stability, atractivity, bifurcation, stability chart and bifurcation chart.

The stability is a property of a dynamic system that characterizes the phase portrait from the vicinity of an invariant set (and not that of an arbitrary set from the phase space) compared to little perturbations of some or of all the dates of the system. By dates we mean all the variables that on which depends the function dynamic system except for the variable independent of time. The most studied stability type is the Leapunov stability, where the perturbations refer only at the initial date. There are stability charts that refer to the change of the cardinal of the stationary set while the last regard only the general qualitative changes of the phase space.

The supermathematical functions are widely discussed in the paper of Șelariu (2007). Any periodical function can describe, obviously, an oscillatory movement. The centric circular functions (FCC) cosine (cosα) and sine (sinα) centric describe, without any difficulty, the oscillatory movements of the liniar systems; the centric mathematics (MC) being the mathematic of the ideal systems, perfect, liniar, while the excentric mathematic (ME) is the mathematic of the real, imperfect, nonliniar systems (Șelariu, 2007). Their reunion reprersent supermathematics (SM).

Some supermathematical circular excentric functions, of centric variable cexθ and sexè and those of centric variable Cexα and Sexθ, are already proven as solutions of some vibrant systems of elastic static nonliniar characteristics. And the quadrilobic functions have proved to be solutions of some oscillatory nonliniar systems, similar with the static, elastic characteristics of the systems of Duffing ($a \times x \pm b \times x^3$), that represent a serial power development, but the quadrilobic vibrations offer a term ($\pm c \times x^5$) of the plus development.

The supermathematical functions realise a vast extension (farther) of the periodical functions, without replacing the unit/trigonometrical circle with closed curves, as we know a lot of tests in the centric mathematic (MC) shortly described in the cited paper. The new periodical functions, component of the mathematical objects and of the eccentric mathematic (ME), which reunited with MC form what we called the supermathematic (SM = MC $\cap$ ME), have already proved as solutions of the description of some periodical movements of some nonliniar vibrant systems, more difficult to describe with the existant functions in MC.

2. Trajectories in the phase plane

In ME it is operated with three angles or variables: at the centre α, at the ex-center θ and the angle β with the centre on the unit circle, in the point W, if the direction ε of expulsion of the ex-center S(s,ε) of the center O(0,0) is on the direction of the axis of abscissa ε = 0, as it will be possible from now on. The angle β can be expressed depending on the centric variable β(α) or the ex-centric variable β(θ). In the present paper there are considered the functions of ex-centric variable θ and the oscillatory system will be called β(θ) and it will be noted with (SOβ).

We have the angular oscillations of the angle β(θ) = bexθ given by FSM-CE

$$\begin{cases} x = \beta \times \mathrm{cex}\theta = \arcsin(s, \cos\theta) \\ y = \beta \times \mathrm{sex}\theta = \arcsin(s, \sin\theta) \end{cases}$$

$$s \in [-1,+1] \Rightarrow x, y : R \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \qquad (1)$$

Of the same fix ex-center S{s,ε}, so that s = ε = constant, where the half-line d^+ rotates around the ex-center S(s,0) with the constant angular speed Ω = 1 rad/s, meaning θ = Ω × t and $\dot{\theta} = \Omega = 1$ and $\ddot{\theta} = 0$.

The oscillation speeds of the angles x, y = f[β(Ω × t)] are:

$$\begin{cases} \dot{x} = \dfrac{d(\beta \times cex\theta)}{dt} = \dfrac{d\theta}{dt}\dfrac{d(bex(\theta - \frac{\pi}{2}))}{d\theta} = \\ = \Omega \dfrac{d(arcsin(s \times cos\,\theta))}{d\theta} = \\ = -\Omega \dfrac{s \times sin\,\theta}{\sqrt{1 - s^2 \times cos^2\,\theta}} = -\Omega \times s \times siq\theta \\ \dot{y} = \dfrac{d(\beta \times sex\theta)}{dt} = \dfrac{d\theta}{dt}\dfrac{d(bex\theta)}{d\theta} = \\ = \Omega \dfrac{d(arcsin(s \times sin\,\theta))}{d\theta} = \\ = \Omega \dfrac{s \times cos\,\theta}{\sqrt{1 - s^2 \times sin^2\,\theta}} = \Omega \times s \times coq\theta \end{cases} \qquad (2)$$

In the expressions of the derivatives/ speeds are distinguished FSM-CE cosine (coqθ) and sine (siqθ) quadrilobe, introduced in mathematics and widely presented in the paper of Șelariu (2007), together with the closed curves called trilobe, quadrilobe and n-lobe or polilobe.

The accelerations of the oscillatory movements are obtained as the second derivative of the oscillatory angles or prime derivative of the oscillatory speeds, this way:

$$\begin{cases} \ddot{x} = -\dfrac{\Omega^2 \times s(1 - s^2)\cos\theta}{(1 - s^2 \times cos^2\,\theta)^{1,5}} \\ \ddot{y} = -\dfrac{\Omega^2 \times s(1 - s^2)\sin\theta}{(1 - s^2 \times sin^2\,\theta)^{1,5}} \end{cases} \qquad (3)$$

The dynamic of the system is given by the differential equation:

$$\ddot{z}(t) = -\Omega^2 \times s(1 - s^2) \times \left[C_1 \frac{cos\,\theta}{(1 - s^2 \times cos^2\,\theta)^{1,5}} + C_2 \frac{sin\,\theta}{(1 - s^2 \times sin^2\,\theta)^{1,5}}\right] \qquad (4)$$

Because the elongations differ pregnantly of the movements from the circular excentric movement (MCE), the integral curves from the phase plan that are expressed depending on the elongations differ pregnantly from those of the circular ex centric movement (MCE).

In the chart 1 are presented curves from the phase plan, but for the solution z = C_1 × x + C_2 × y, with C_1 = C_2 = 1, and in figure 3 are presented CES which, as it can be noticed, are very much non-linear for amplitudes of the large elongations, passing into branches with negative rigidity.

From the figure 1.b where the integrale curves from the phase plane are represented in 3D, it can be noticed that they are the same both for s ∈ [-1, 0] and for s ∈ [0, +1].

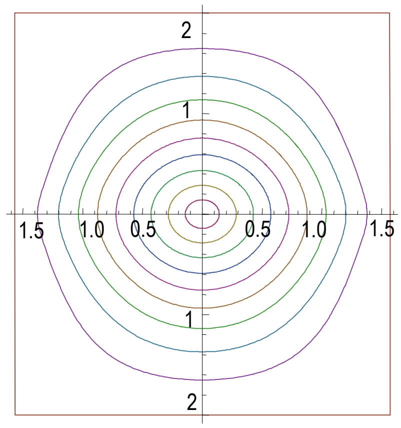


Figure.1.a. *Integrale curves in the phase plane*
Trajectories of phase $\dot{x}(x)$ *și* $\dot{y}(y)$, $C_1 = C_2 = 1$

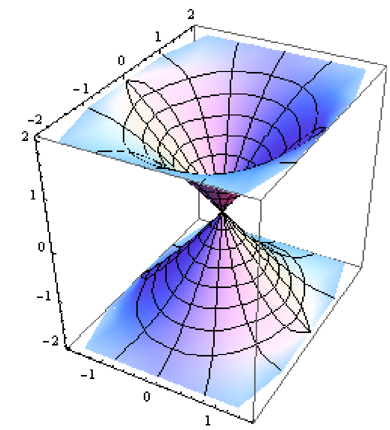
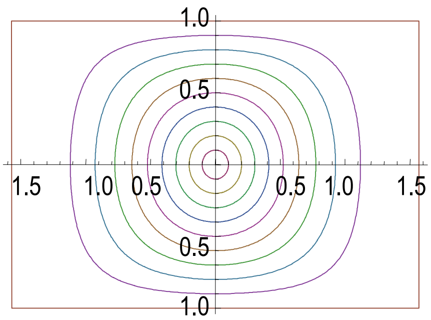
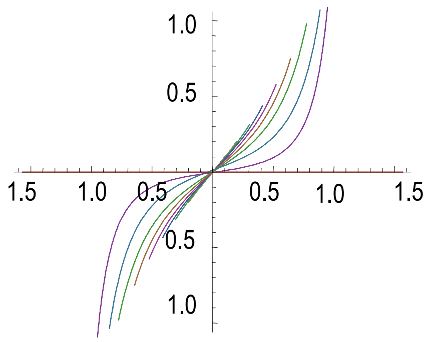


Figure 1.b. *Integrale curves in the phase plane*
Trajectories of phase $\dot{x}(x)$ *și* $\dot{y}(y)$ *in 3D,* $s \in [-1,1]$

In figure 2 are restored the integrale curves from the phase plane for each oscillation and x and y separately, their curves being identical. From the figure 1.a it can be noticed that for the excentricity s = e = 0 the oscillation stops, and for low values of the s excentricity or e integrale curves from the phase plane approach the circle, meaning linear oscillations, while for high values of the eccentric numeric s or the real one e the oscillations are strongly non-linear, what can be better noticed in the figure 2b directly on CES that are, for SOβ, CES strong or progressive.





a) Trajectories of phase

b) Elastic static characteristics

Figura 2. *The speeds (trajectories of phase) and the accelerations (elastic static characteristics (CES))*
As functions of oscillation elongation x(t) and, respectively, y(t)

The importance of the oscillation study SOβ consists in their resemblance with free vibrations of the systems with CES Duffing (Fel = k × x - β × x³), for β < 0 .

Developing in power series one of the functions that express the elongation SOβ, as an example, y = arcsin u = arc sin[s × sinθ] is obtained:

$$y = arcsin u = u + \frac{1}{2\times 3}u^3 + \\ + \frac{1\times 3}{2\times 4\times 5}u^5 + \frac{1\times 3\times 5}{2\times 4\times 6\times 7}u^7 + \ldots = \\ = u + 0.166666u^3 + 0.00297619u^7 + \ldots \qquad (5)$$

And it is noticed that the first two terms are identical with those of CES Duffing for k = 1 (x + β × x³).

Replacing u with the expression $u = s \times \sin\theta = \sin\beta(\theta)$ it is noticed that if $s \in [-1,+1], \theta \in [0, 2Pi] \rightarrow \beta(\theta) \in [-\pi/2, +\pi/2] \rightarrow u \in [-1, +1]$ and it is assumed that the vibration takes place for subunitary u amplitudes.

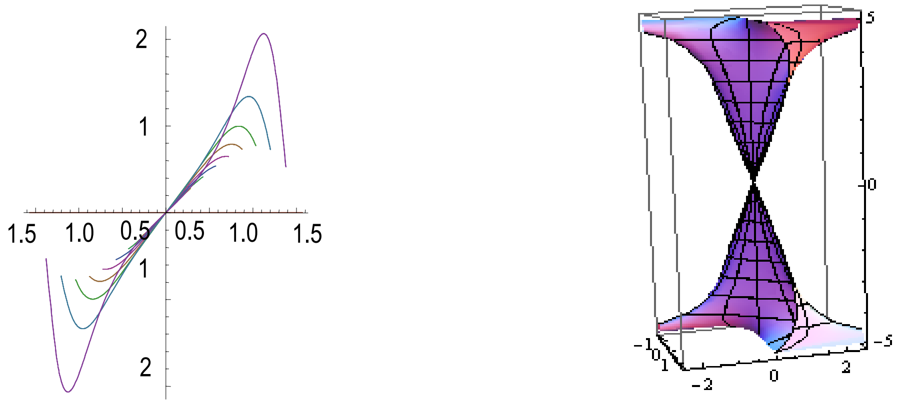


Figure 3. *The acceleration depending on the z oscillation* $\rightarrow -\ddot{Z}(Z)$, *for* $C_1 = C_2 = 1$, *in 2D and in 3D*

Further on are presented three dynamic systems with applications in economy for which the stability can be studied with the help of the method of the Leapunov function.

1. The model of urban dynamic of type Lorenz is described by the system of differential equations (5).

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(a_2 \times x_2 - a_3 \times x_1) \\ \dfrac{dx_2}{dt} = c_1(c_2 \times x_1 - c_3 \times x_2) - c_4 \times x_1 \times x_3 \\ \dfrac{dx_3}{dt} = d_1 \times x_1 \times x_2 - d_2 \times x_3 \end{cases} \quad (5)$$

The economical measures that appear are: x_1-the production of the urban system, x_2-the number of residents, x_3-the rent land and a_i, c_i, d_i are positive paramethers.

2. The evolution model of the capital of a firm is described by the system of differential equations (6).

$$\begin{cases} \dfrac{dx_1}{dt} = c \times x_1^2 \times x_2 + b \times x_1 \\ \dfrac{dx_2}{dt} = x_1 + a \times x_2 - 1 \end{cases} \quad (6)$$

$$x_1 = x_1(t), x_2 = x_2(t)$$

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The economical measures that appear are:

x_1 - the capital of the firm and x_2 - the volume of the work force.

3. The simple model of Ramsey type is described by the differential equation (7).

$$\frac{d^2z}{dt^2} = \frac{dz}{dt}\frac{dq}{dt} + \left(\frac{dq}{dt} - p\right)\frac{U'}{U''} \quad (7)$$

The economical measures that appear are: U-utility of the consumption, c-the consumption, z-the proportion capital-work, $\frac{dz}{dt}$-the installment of the capital accumulation, p-the decided installment of the discount, $p \geq 0$. We use the following notations:

$$c = q(z) - \frac{dz}{dt}$$

$$U' = \frac{dU}{dt}, \; U' > 0$$

$$U'' = \frac{d^2U}{dt^2}, \; U'' < 0$$

For the study of the stability of the systems of differential equations given by (5), (6) and (7) can be succesfully used original methods of the author presented by Bălă (2004).

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