The Impossibility of a Paretian Libertarian: a Solution through the Imposition of a Rationality Condition on Individual Preferences on Conjunct Alternatives

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Abstract. *In this article I present the Impossibility of a Paretian Libertarian theorem (Sen) and I discuss its significance with regard to the problem of externalities. I introduce two conditions for the selection of an "acceptable" solution for this impossibility result: efficacy for real societies and ideological fidelity. I present Hillinger-Lapham, Gibbard, Blau and Saari solutions and I classify them according to the two conditions mentioned above. I extend the social choice literature by introducing a new solution consisting in a reformulation of the libertarian at the minimal-rational libertarian preferences. In this formulation the only relevant kind of preferences are those that are "minimal-rational libertarian". This solution satisfies both the efficacity for real societies and the ideological fidelity conditions.*

Key words: externalities; ideological fidelity; social welfare function; social decision function; minimal-rational libertarian preferences.

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1. Social choice theory (SCT)

In 1951, Kenneth J Arrow has set the foundations of the axiomatic Social Choice Theory[1], through his General Possibility Theorem. This result showed that there is no social welfare function which satisfies a few reasonable conditions: unrestricted domain, independence of irrelevant alternatives, non-dictatorship and weak Pareto principle. The significance of Arrow's theorem was, in a strong interpretation, either the welfare economics is consistent only with dictatorship, either democracy it's not possible. In a weak interpretation one (or more) condition is not necessary and must be weakened or eliminated. Nineteen years after Arrow's paradigmatic result was published, Amartya K. Sen had proposed the impossibility of a paretian liberal theorem. Through this impossibility result, Sen introduced in the social choice theory research the problem of individual rights. This new result appears due to the 1) relaxing the welfare function to a social decision function (from social transitivity to social aciclicity) 2) introducing the individual rights problem in social choice theory's concerns, through a libertarian condition, 3) retaining universal domain and weak pareto conditions, 4) eliminating nondictatorship and independence from irrelevant alternatives conditions. The importance of Sen's theorem is, in a strong interpretation, that the foundations of welfare economics, the paretian principle, is not compatible with individual rights and minimal social rationality. In a weak interpretation, Sen's impossibility result does not express an inconsistency between paretianism and individual rights, but an inconsistency between paretianism and a certain formulation of individual rights, formulation which may be changed into another that is consistent with minimal rationality and paretian principle. These two impossibility results, Arrow's and Sen's theorems, have generated a vast literature in many different paradigms and disciplines. Alternative demonstrations and solution have been proposed by mathematicians, philosophers, but especially by economists.

In order to consider these solutions as part of Social Choice Theory, in its classic form, a few conditions must be met. I will briefly enounce it. First, all Social Choice Theory approaches are axiomatic, meaning that the heuristic basis, on which the entire system is built, depends on the acceptance of various axioms. Second, all Social Choice Theory approaches are methodological individualistic. This characteristic of *SCT* says that the fundamental research unit is the individual. In order to explain social action or choice we must precede analytically, studying individual action or choice and the ways in which these individual actions or choices are aggregated into social action or social choice. Therefore the only fertile form of social research is that which explains society starting from individuals. Third, all *SCT* approaches are welfarist in nature, meaning that the relevant information set is limited to information about preferences. Starting from these preferences we can define two rather weak forms of individual rationality. An individual is considered rational when his preference relation is reflexive, complete and transitive. In the same way, an individual is considered minimal-rational if his preference relation satisfies reflexivity, completitude and aciclicity. Fourth, all *SCT*

research are a priori in nature. Last, the primitive of arrowian *SCT* is a binary preference relation, formalized in trivalent logic. This preference relation is denoted by R and its name is "weak preference relation between alternatives". I will present now some of the notations, definitions and theorems in a form which resembles that of Sen[2] in (1970). The notations are to be read in the following way: *d=definition*, *a=axiom*, *t=theorem*.

d1 (weak preference relation): x is weakly preferred to y if and only if either x is strictly preferred to y, either y is strictly preferred to x: $xRy \leftrightarrow (xPy \vee xIy)$;

d2 (strict preference): x is strictly preferred to y if and only if x is weakly preferred to y and y it is not weakly preferred to x: $xPy \leftrightarrow (xRy) \wedge \neg(yRx)$;

d3 (indifference relation): x is indifferent to y if and only if x is weakly preferred to y and y is weakly preferred to x: $xIy \leftrightarrow (xRy) \wedge (yRx)$;

a1 (reflexivity): x is in relation of weak preference with himself: $\forall x \in A : xRx$;

a2 (completitude): for all different alternatives x, y, either x is weakly preferred to y, either y is weakly preferred to x: $\forall x, y \in A : (x \neq y) \rightarrow (xRy) \vee (yRx)$;

a3 (transitivity): for any 3 alternatives, if the first is weakly preferred to the second and the second to the third, then the first must be weakly preferred to the third: $\forall x, y, z \in A : (xRy) \wedge (yRz) \rightarrow xRz$;

a4 (aciclicity): for any finite number of alternatives, if the first is strictly preferred to the second, the second to the third and so on, then the first must be weakly preferred to the last: $\forall x, y, z, ..., v \in A : (xPy) \wedge (yPz) \wedge, ..., \wedge zPv \rightarrow xRv$;

d4 (collective choice): a collective choice rule (*CCR*) is a functional relation f such that, for any set of n individual orderings (transitive, complete and reflexive individual preference relations), $\mathcal{R}_1, ..., \mathcal{R}_n$ *(one ordering for each individual), one and only one social preference relation R is determined,* $R_s : f(\mathcal{R}_1, ..., \mathcal{R}_n)$;

d5 (choice set): an alternative x from A *is the best alternative in A with regard to a binary weak preference relation, if and only if it is at least as good as any other alternative in A:* $\forall y : \big((y \in A) \rightarrow (yRx)\big)$. *The set of the best alternatives in A is called choice set of A and it is denoted with* $C(A, R)$;

d6 (minimal collective rationality): a social decision function (SDF) is a collective choice rule f, the range of which is restricted to those preference relation R, each of which generates a choice function C(S,R) over the whole set of alternative X. i.e. reflexive, complete and acyclic social preference relations;

d7 (collective rationality): a social welfare function (SWF) is a collective choice rule f, the range of which is restricted to the set of orderings for society. i.e. reflexive, complete and transitive social preference relations;

t1 (non empty choice set): $C_s(A, R) \neq \varnothing$, if R is reflexive, complete and acyclic. *t1* is proved by Sen in (1970a);

t2 $\text{FBS} \rightarrow \text{FDS}$. *i.e. any social welfare function is also a social decision function but not vice versa. From here we must conclude that Sen's result has also validity for social welfare functions. t2* is proved by Seidl in (1975).

2. The impossibility of a paretian libertarian

In this paper I will analyze Sen's result[3] (1970a, 1970b). Therefore, I will focus my at-

tention on the inconsistency between weak paretianism, individual rights and universal domain. First I will define the conditions and then I will present Sen's demonstration for the theorem.

Condition L (libertarianism): for each person i, there is at least one pair of distinct alternatives (x,y) such that he is decisive in the social choice between them in either order: if I prefers x to y then society will prefer x to y and if i prefers y to x, then society will prefer y to x: $D_i(x,y) \wedge (xP_iy) \rightarrow xP_sy$ *and* $D_i(x,y) \wedge (yP_ix) \rightarrow yP_sx$.

*Condition L** (minimal libertarianism): *ceteris paribus L, but the set of individuals has a cardinality of 2.*

Condition U (unrestricted domain): a social preference function has an unrestricted domain if it admits all and only individual orderings (reflexive, complete, transitive preference relations).

Condition P (weak pareto): for any two alternatives x, y, if all individuals strictly prefer x to y, then x is socially preferred to y: $(\forall i : xP_iy) \rightarrow xP_sy$.

t3 (the impossibility of a paretian libertarian): Sen, (1970a, 1970b). There is no SDF which satisfy U, P, L\L,when the number of individuals is at least 2 and the number of alternatives is at least 3.*

Proof: from the premises $N = \{i, j\}$, and $A = \{x, y, z, w\}$. Suppose that there is a *SDF* which satisfy *U, P, L**. We must check three cases: 1). *(x,y)* and *(z,w)* are the same pair. It is obvious, form *L** that we can't give libertarian decisivity for two individuals on the same pair of alternatives, because the choice set would be empty. 2) The pairs have one of the elements in common. Suppose that $x = z$, $xP_iy \wedge yP_iw$, and $wP_jx \wedge yP_jw$. By condition *L**, $D_i(x,y)$ and $D_j(x,w)$, by *P,* we have yP_sw, by *U*, any individual rational preference is relevant, and by aciclicity the social choice set must be non empty. Therefore $xP_sy \wedge yP_sw \wedge wP_sx$, but $wP_sx \leftrightarrow \neg[xRw]$ which means that the aciclicity condition is violated and there is no best alternative. 3) Suppose now that all four alternatives are distinct. Assume that $wP_ix \wedge xP_iy \wedge yP_iz$ and $wP_jx \wedge yP_jz \wedge zP_iw$. By condition *L**, $D_i(x,y)$ and $D_j(z,w)$, by *P*, $wP_sx \wedge yP_sz$, by *U*, any individual rational preference is relevant, and by aciclicity the social choice set must be non empty. But $xP_sy \wedge yP_sz \wedge zP_sw \wedge wP_sx$, and $wP_sx \leftrightarrow \neg[xRw]$, though by aciclicity xRw, which means that the aciclicity property is violated and there is no best alternative. The proof works for all cases in which we give one libertarian decisivity to each individual.

Sen (1970a, 1970b) offers an example for his impossibility result.

Lewd vs. Prude case: suppose we have two individuals, *Prude* and *Lewd*, and they must socially decide the reading of a copy of D.H. Lawrence's, "*Lady Chatterly's Lover*". The alternatives are: $x : (1,0)$, *Prude* reads the book and *Lewd* doesn't read it, $y : (0,1)$, *Prude* doesn't read the book and *Lewd* reads it; $z : (0,0)$, nobody reads the book. Suppose $zP_px \wedge xP_py \wedge zP_py$; meaning that *Prude* prefers most that no one reads the book, next that he reads it, and last that impressionable *Lewd* be exposed to obscene literature. Suppose also that $xP_ly \wedge yP_lz \wedge xP_lz$. i.e. Lewd prefers that either of them should read the book, rather than neither, but further he prefers that prude should read the book rather than himself. By U, all individual orderings are admissible. By condition *L**, $D_p(x,z) \wedge (zP_px) \rightarrow zP_sx$. In the

same manner, $D_l(y,z) \wedge (yP_lz) \rightarrow yP_sz$; by ***P,*** $xP_py \wedge xP_ly \rightarrow xP_sy$; *by A:* $xPy \wedge yPz \rightarrow xRz$, deci, $zPx \leftrightarrow \neg[xRz]$. Though, from *P*,*L** and *U* we have

$D_p(x,z) \wedge (zP_px) \rightarrow zP_sx$,

$D_l(y,z) \wedge (yP_lz) \rightarrow yP_sz$ and

$xP_py \wedge xP_ly \rightarrow xP_sy$, so $xPy \wedge xPy \wedge zPx$; but from *A*, $zPx \leftrightarrow \neg[xRz]$. It is clear that acyclicity is violated and the *Prude vs. Lewd* case is an ilustration of the impossibility of a paretian libertarian.

3. The problem of externalities and the ideological fidelity criterion

As we can easily see in the *Lewd vs. Prude* case, each individual has strict preferences on the other's behavior. If we assume a stronger meaning of individual rationality, we can understand the problem in the following way: suppose that an individual is rational not just because he has minimal-rational preferences (acyclic) or rational preferences (transitive), but because his preferences have a direction. Therefore, we can assume that an individual will prefer more of a good rather than less and that he will be indifferent between identical quantities of identical goods. From this we can assume that any strict preference for a good (or alternative) shows a difference in the level of satisfaction produced by that good. If we conceive the behaviors of other people as parts of alternatives that may be socially preferred (*standard TAS assumption*), and if we give individuals the right to determine the social preference between some alternatives which are personal to them (the libertarian condition), then if we observe that the individuals are not indifferent between alternatives which are personal to other individuals, we can deduce that they are, in some way, affected by the behaviors of others. In the terms of consumption and productivity theory, and given the condition mentioned above, the problem is that the outputs of an economic agent are inputs in the productivity or the consume vector of another economic agent. We know that standard SCT doesn't allow negotiations and side payments and given this knowledge we can say that the impossibility of a paretian libertarian may be treated as a problem of externalities. Any time we observe a strict preference of an individual on alternatives which are different regarding the behavior of another individual, we can deduce the existence of externalities[4]. This problem shouldn't surprise us too much because in 1962 Buchanan and Tullock offered a proof that any collective decision which is not produced by unanimity rule may produce externalities. Hence, in the impossibility of a paretian libertarian, we do have two rules one of which is subunanimous (the libertarian condition) we have to expect externalities. In other words, if we use a non paretian rule we permit externalities to appear. Also, in 1982, Bernholz[5] offered a proof of the fact that externalities are a necessary condition for the apparition of social cyclic preferences. (*more precise, intransitive and cyclic*) in non-dictatorial societies. The same thing may be observed in another way: as long as 1) the individual preferences are, by U, transitive and complete, 2) the alternatives contain other's actions or behaviors, then, if we

observe any another thing than indifference between those alternatives that varies with respect to another individual's behavior, we can deduce the existence of externalities. The argument is built in the following manner: if the externalities had been absent, then *Lewd* and *Prude* would have been indifferent on any pair of alternatives considered personal to the other. This means that on any pair personal to someone, we would have a weak social preference in the direction of the decisive individual's preference. To clarify this idea: if I have the right to read or not to read a book and there are no externalities, you will be indifferent if I read it or not. From this, the social preference on such pairs will be given by the strong paretian procedure. *i.e.* at least on individual have a strict preference for an alternative, the others being indifferent[(6)]. But in the case of the impossibility of a paretian libertarian theorem we do observe that each individual has strong preferences on alternatives personal to others! This must have the meaning mentioned above: the impossibility of a paretian libertarian is a case in which externalities are present.

Once we understood the problem as one of externalities, we must decide what to do next: 1) are all externalities relevant? 2) What should we do if we consider that there are all relevant? What should we do if we consider that not all externalities are relevant? Regarding the first question, we have either the possibility to consider that the decision to read or not to read a book is absolutely personal to individuals, either to consider that each and every time when an externality is present it must be considered relevant. In the first case we give priority to individual rights which offers strong protection exclusively on those alternatives that varies regarding one individual's behavior. In the second case we have a choice between reducing to silence the individual rights in some level conditioning them in a way that avoids aciclicity. In this last case we give priority to the paretian criterion which offers a weaker protection than the libertarian condition but the domain of this protection is much bigger – extended to all of the alternatives. In practice, these solutions have either *1)* the libertarian form in which the state, either doesn't exist, either it has no right to interfere; 2) the classic form of welfare economics in which the state interfere in order to compensate the loss produced by negative externalities. This problem can not be scientifically solved. The option, which ever would that, be it's in ethics domain. I will assume a libertarian position and I will consider, in a lockean manner, that the individual it's the only owner of his body and of its faculties. From here, we can draw a personal sphere for each individual and this sphere can not be alienated on the base of any externalities based argument. The fact that I am learning hard for an exam may cause insecurity to another competitor less available for making an effort. This feeling is, obviously, an uncompensated loss. However on the basis of self ownership principle nobody can force me to invest my effort in other ways. It is irrelevant that my effort investment makes others to feel threaten. If we consider that this is the foundation for individual rights we are forced to admit the idea of libertarian

partition of which Farrell (1976) talks among others: there is a set of human actions which are considered personal to individuals. The externalities produced by these actions are considered irrelevant. This is what I will call the ideological fidelity criterion. As long as we whish to formalize a libertarian condition, we must make an ethical hierarchy: the libertarian condition must be considered more important than any other condition we use. Starting from this idea, we must impose this criterion to any solution for the inconsistency discovered by Sen. Next to this criterion I will enounce another one the scope of which is the elimination of inconsistencies for real societies.

4. Efficacity condition

Sen's formulation of the libertarian condition is ambiguous, giving libertarian decisivity for "at least one pair" of alternatives which vary in what concerns the behavior of only one individual. It is not told what "at least means", although Sen gave a decisivity for each person. In 1977 Breyer and Heidelberg named this condition "issue liberalism" and they eliminated the ambiguity, giving exactly one decisivity for each individual. In 1976, Sen gives the impression that he assimilates this idea of defining on issues although he doesn't formalize it. For a better understanding I will introduce and define the notion *x – aspect* and reengage the definition given by Gibbard (1974) and Breyer (1977) to the notion of *x – variant* (*the formalization in mine*):

d8 (*x – aspect's* set): $\forall x, y, z \in A, \forall i, j \in N, \quad x=(x_i, x_j), y=(y_i, y_j), z=(z_i, z_j),$ then $X_a = X_{ai} \cup X_{aj}$ is the set of the *x –aspect's* of A, regarding the individuals *i* and *j*, where $X_{ai} = \{x_i, y_i, z_i\}$ and $X_{aj} = \{x_j, y_j, z_j\}$.

d9 (*x – variant's* set): for $\forall x, y \in A, \forall i, \quad j \in N, \quad x=(x_i, x_j), \quad y=(y_i, y_j),$ if, $(x_i \neq y_i) \wedge (x_j = y_j)$ then *x* and *y* are *i – variants*; in the same way, if $(x_i = y_i) \wedge (x_j \neq y_j)$, then *x* and *y* are *j – variants*. The set of *x – variants* equals the set of *x – aspects* of *A* from which we substract the set of *x – aspects* which doesn't vary in the way previously defined: $X_v = X_a \setminus X_{nv}$. Where X_{nv} is the set of *x - aspects* which are not *x – variants*.

In Sen (1970a, 1970b), Breyer (1977), Gibbard (1974), Gaertner and Kruger (1981), (1983), Blau (1975), Suzumura (1978), Austeen-Smith (1982) consider that decisivities are not granted on all *x – variants* but just on one pair for each individual. In 1989, Gardenfors and Pettit gave an example in which the number of decisivities equals the number of *x – variants*, but they didn't propose this as a general rule of granting decisivities. The problem is that there is no ethical reason to justify limiting the number of decisivities to one for each of individuals. As long as we accept that it is in the libertarian spirit to grant individual rights on the basis of *x – variants*, there is no reason to use this on any pairs of. If we use this on one pair of alternatives, we can't give a good reason not to extend this on all pairs which have the property of being *x – variants*. For these reasons the rule for granting individual rights must be the following:

d10 (Rdl: the rule for granting libertarian decisivities): $\#D_x = \#X_v$. *In words: the cardinality of the set of libertarian decisivities equals the cardinality of the set of x – variants.*

The consequence of this rule is that when we want to give a solution for Sen's impossibility result, we must keep in mind that it must resolve all the cases in which $\#D_x = \#X_v$. Next to this rule all solutions of the inconsistency must solve the problem for real societies i.e. societies with any finite number of individuals and alternatives. This is the efficacity criterion. In the following line I will present four solutions for Sen's result and I will analyze them with the tools of the two criterions already stated in sections 3 and 4: ideological fidelity and efficacy for real societies.

5. Solutions for the impossibility of a paretian libertarian theorem

In 1984, Sen showed that if we want to solve the inconsistency between minimal rationality, weak pareto condition, unrestricted domain and libertarianism, there are at least two ways: either we give up to one of the conditions; either we choose to weaken one of them in a substantial way. Regarding the unrestricted domain, Sen adds that any weakening of this condition implies the elimination of some individual preference profiles. In 1976, 1983, Sen argued against this kind of solution because restricting the unrestricted domain is a recognition of defeat. I will assume this position and I will present three restrictions of the libertarian condition and one of the weak pareto condition.

When the problem of altering one of the conditions which produced the libertarian inconsistency was addressed, the libertarian condition was, usually, the first target of SCT researchers. I will present only three of such solutions. The first was proposed by Hillinger and Lapham (1971). Their solution is a strong weakening of the libertarian condition which makes individual rights to be a form of paretianism. An individual is decisive on a pair of alternatives if and only if there is nobody who opposes his preference on that pair:

Condition L_{hp}*:*

$$\forall (x,y), x \neq y, \exists i \in N : \exists D_i(x,y) \leftrightarrow$$
$$[((xP_iy) \wedge \neg[yR_{N-i}x]) \vee ((yP_ix) \wedge \neg[xR_{N-i}y])]$$

Another solution of Sen's result is the alienable rights solution introduced by Gibbard in 1974. *Condition* L_{al}: *for all individuals i, if i is decisive on a pair of x - variants (x,y) and i prefers x to y, then, usually, x will be socially preferred to y. But if there is an alternative z, and i prefers y to z, and there is an individual j who is decisive on the pair of x - variants (z,x) and j prefers z to x, the right of i on (x,y) is waived*: L_{al}: $\forall x,y,\forall i,j, D_i(x,y) : A_i(D_i(x,y)) \leftrightarrow \exists z, yR_iz \mid D_j(z,x) \wedge zR_jx$, where $A_i(D_i(x,y))$ must be read in the following manner: "*the right of individual i on the pair (x,y) it is waived*". This solution generates a social preference equivalent to weak pareto preference. An approach similar to Gibarrd's, is that formulated by Blau in 1975. His solution consist in modifying the libertarian condition in order to eliminate a certain kind of preference, intrusive preferences. These are defined in the following way: "*an individual i is intrusive if he is not indifferent between the alternatives from his own personal sphere* D_i*, and, for some j, his opposition against j's preference on* D_j *is stronger than his own preference on* D_i*. From this definition Blau defines three libertarian*

conditions: Condition L_{ni1}: if all individuals have intrusive preferences then all decisivities are waived. Condition L_{ni2}: if some individuals have intrusive preferences then all decisivities are waived. Condition L_{ni3}: all decisivities of those individuals with intrusive preferences are waived.

I will now present Saari's (1997) solution by restricting the pareto condition. His argument has two parts: First, there is an incompatibility between the separability dimension between pairs of the libertarian condition and the connectivity between them presented by the pareto condition. Second, in Saari's own words, "*if society grants me the right to choose my shirt, why are others comparing one of my alternatives with other alternatives?*"(Saari, 1997, p. 92). This suggests relaxing weak pareto condition to: "if an individual is given decisive rights over a pair *(x,y)*, then the pareto condition P does not apply to any pair including either x or y" (Saari, 1997, p. 92).

I will analyze now, briefly, all four solutions on the basis of the ideological fidelity criterion and on the efficacity for real societies criterion. The Hillinger-Lapham solution satisfy the efficacity condition through the transitivity of the strong pareto condition, but doesn't satisfy the ideological fidelity criterion because it silent the libertarian condition. The Gibbard solution doesn't grant the libertarian decisivities according to rule. The same problem is present in the case of Blau's solution. Another problem of this last solution is that its works only for two individuals and two alternatives. Neither of these two solutions satisfy the ideological fidelity criterion because they waive the individual rights and not the pareto condition. Finally Saari's condition satisfies both criterions. Since the space is limited I will ask the reader to check these results.

6. Minimal-rational libertarian preferences

I introduce a new solution for Sen's inconsistency. This solution satisfies the ideological fidelity and the efficacity criterions. *i.e.* it works for the *Rdl* rule and it operates an ethical hierarchy in favor of individual rights. I will give now some definitions.

d11 (simple alternatives): a simple alternative refers to the action of a single individual. e.g. x: "I read the book" (1); y: "I don't read the book" (0).

d12 (conjunct alternatives): a conjunct alternative refers to the action of two or more individuals. e.g. x: "I read the book and you don't read it" (1,0), y: "I don't read the book and you read it" (0,1), z: "I don't read the book and you don't read it" (0,0).

d13 (choice dimension): a choice dimension regarding an individual i is a list of simple alternatives: $\Delta_i = \langle x, y, ..., n \rangle \mid \Delta_i \geq 2 < \infty$. *e.g. for any two simple alternatives "I read the book" (1) and "I don't read the book" (0), together forms the choice dimension of reading the book regarding individual i.*

d14 (conjunct alternatives set[7]): $\# A_c = n^k$, *where* A_c *is the set of conjunct alternatives, n is the number of simple alternatives and k is the number of individuals.*

d15: the preferences of an individual i are libertarian minimal-rational if and only

if $\forall i,j,(i \neq j),\ \forall A, A=\{x,,y,z,...,n'\}$, *with x – aspects:* $x=(x_i,x_j)$, $y=(y_i,y_j)$, $z=(z_i,z_j)$, $n'=(n'_i,n'_j)$, *then* $C(\{xi,yi,zi,...,n'_i\},R)\neq\varnothing$ *for any* $x_j,y_j,z_j,...,n'_j$. *i.e. i must have an unempty choice set on the set of his x – aspects.*

From *t1* we know that if aciclicity and completitude are sufficient and necessary for having an unempty choice set. From here, *d12* says that all individuals must have a complete and acyclic preference on his own *x – aspects*.

Condition L_{mr} *(minimal rational libertarianism): for any finite number of individuals, the libertarian decisivity is granted on the basis of Rdl:* $\#D_x = \#X_v$ *rule, only to the individuals who have libertarian minimal-rational preferences.*

t4 (the possibility theorem of minimal rational libertarianism) for any n ≥ 2 individuals, and for any dimension (with one or more simple alternatives), there is a SDF defined over U which satisfies P and L_{mr}.

Proof: in order to prove this theorem it must be shown that: *1)* L_{mr} works for two individuals and two dimensions (*one for each individual*), each containing two simple alternatives; *2)* L_{mr} works for two individuals, two dimensions with three or a higher number of simple alternatives for each individual; *3)* L_{mr} works for three or a higher number of individuals and for two dimensions with two simple alternatives and *4)* L_{mr} works for three or a higher number of individuals and for two dimensions with two or a higher number of simple alternatives.

Case 1: suppose there are two individuals *i, j* and two dimensions denoted here by *(0,1)*, for simplicity. We have, therefore, $\Delta_i=\langle 1,0\rangle$ *and* $\Delta_j=\langle 1,0\rangle$. From *d11*, $\#A_c=4$ and A_c=*{(1,1), (1,0), (0,1), (0,0)}*. We denote by $x=(x_i,x_j)$, $y=(y_i,y_j)$, $z=(z_i,z_j)$, $w=(w_i,w_j)$, where $X_{ai}=\{1,1,0,0\}$, and $X_{aj}=\{1,0,1,0\}$. Suppose now that *i* and *j* have *minimal rational libertarian preferences* over *A*. By aciclicity and by completitude of his preference relation, each individual has an unempty personal choice set. We must verify now if there is an unempty choice set. In order to succeed, we must determine all individual decisivities with the help of L_{mr} condition. Therefore, we have $D_i(y,w)\wedge D_i(x,z)$ and $D_j(x,y)\wedge D_j(z,w)$. We have to verify four cases: $0P_i1\wedge 0P_j1$, $0P_i1\wedge 1P_j0$, $1P_i0\wedge 1P_j0$, $1P_i0\wedge 0P_j1$. Suppose that $0P_i1\wedge 0P_j1$; in this case $(z_iIw_i)P_i(y_iIx_i)$ and $(y_jIw_j)P_j(x_jIz_j)$, but, by $L_{mr,}$

$(w_iP_iy_i)\wedge D_i(y,w)\rightarrow wP_sy$,

$(z_iP_ix_i)\wedge D_i(x,z)\rightarrow zP_sx$,

$(y_jP_jx_j)\wedge D_j(x,y)\rightarrow yP_sx$,

$(w_jP_jz_j)\wedge D_j(z,w)\rightarrow wP_sz$.

By P, $(w_iP_ix_i)\wedge(w_jP_jx_j)\rightarrow wP_sx$, therefore, $(wP_sy)\wedge(yP_sx)\wedge(wP_sz)\wedge(zP_sx)\wedge(wP_sx)$. From here, $M_s(\{x,y,z,w\},R)=\{w\}$, *i.e.* everybody gets what they wanted *(0,0)*. For the case in which both prefer 1 to 0 the situation is the same. I will address now the second case: $0P_i1\wedge 1P_j0$, therefore $(z_iIw_i)P_i(y_iIx_i)$ and $(x_jIz_j)P_j(y_jIw_j)$. By L_{mr}: $(w_iP_iy_i)\wedge D_i(y,w)\rightarrow wP_sy$,

$(z_iP_ix_i)\wedge D_i(x,z)\rightarrow zP_sx$ and ,

$(x_jP_iy_j)\wedge D_j(x,y)\rightarrow xP_sy$,

$(z_jP_iw_j)\wedge D_j(z,w)\rightarrow zP_sw$.

By P: $(z_iP_iy_i)\wedge(z_jP_jy_j)\rightarrow zP_sy$, so we have: $(zP_sw)\wedge(zP_sy)\wedge(zP_sx)\wedge(wP_sy)\wedge(xP_sy)$. From here, $M_s(\{x,y,z,w\},R)=\{z\}$, i.e. everybody gets what they wanted *(0,1)*. The other case is the same as this one.

Case 2: suppose there are two individuals *i, j* and two dimensions with three simple alternatives for each individual $\Delta_i = \langle 1,0.5,0 \rangle$, and $\Delta_j = \langle 1,0.5,0 \rangle$. From *d11*: $\# A_c = 3^2$, *i.e.* $A_c = \{p.q,r,s,t,u,v,x,y\}$. For simplicity i will use the following table:

Alternatives, x-variantes, decisivities

Table 1

A	cod.	i-asp.	j-asp.	D_i	D_j
p	*(0.5, 0)*	p_i=0.5	p_j=0	(p,v)	(p,s)
q	*(1, 0.5)*	q_i=1	q_j=0.5	(p,x)	(p,t)
r	*(0, 0.5)*	r_i=0	r_j=0.5	(v,x)	(s,t)
s	*(0.5, 1)*	s_i=0.5	s_j=1	(q,r)	(r,u)
t	*(0.5, 0.5)*	t_i=0.5	t_j=0.5	(q,t)	(r,x)
u	*(0, 1)*	u_i=0	u_j=1	(r,t)	(u,x)
v	*(1, 0)*	v_i=0.5	v_j=0	(s,u)	(q,v)
x	*(0, 0)*	x_i=0	x_j=0	(s,y)	(q,y)
y	*(1,1)*	y_i=1	y_j=1	(u,y)	(v,y)

Table 1 is to be read in the following way: *A* is the set of alternatives on the second column. The codification (cod.) is useful in order to distinguish between *x-variants*, and can be exemplified as: (p_i=0) "*i doesn't read the book*"; (q_i=1) "*i reads the book*" and (s_i=0.5) "*i reads half of the book*". Moving on *i-aspects (i-asp.)* represents the aspects of individual i, and *j-aspects* represents those parts of the conjunct alternatives which express individual j's behavior. D_i is the set of libertarian decisivities of individual i, and D_j is the set of libertarian decisivities of individual *j*. We can see in the table the following equivalences: for *i:* p_i,s_i,t_i have all *(0.5)*; r_i, u_i, x_i, have all *(0)*; q_i, v_i, y_i, have all *(1)*; for *j*: q_j, r_j, t_j have all *(0.5)*; p_j, v_j, x_j, have all *(0)*; s_j, u_j, y_j, have all *(1)*. Since each of these equivalences shows the same individual behavior, if we ignore the problem of externalities, we can consider that our individuals must be indifferent between the following alternatives: $p_i Is_i It_i(0.5)$, $q_i Iv_i Iy_i(1)$, $r_i Iu_i Ix_i(0)$, $p_j Iv_j Ix_j(0)$, $q_j Ir_j It_j(0.5)$, $s_j Iu_j Iy_j(1)$. If this doesn't happen, then they don't have minimal rational libertarian preferences. We are interested only in the cases in where one of the real alternatives dominates the other two; we have, therefore, 9 possible: $0.5P_i(0,1) \wedge 0.5P_j(0,1)$, $0.5P_i(0,1) \wedge 0P_j(0.5,1)$, $0.5P_i(0,1) \wedge 1P_j(0.5,0)$, $1P_i(0.5,0) \wedge 1P_j(0.5,0)$, $0P_i(0.5,1) \wedge 0P_j(0.5,1)$, $1P_i(0.5,0) \wedge 0.5P_j(0,1)$, $1P_i(0.5,0) \wedge 0P_j(0.5,1)$, $0P_i(0.5,1) \wedge 0.5P_j(0,1)$, $0P_i(0.5,1) \wedge 1P_j(0.5,0)$. All these cases must be discussed starting from an observation: all decisivities of one individual are in the equivalence sets of the other. In other words, each individual will be indifferent between all the alternatives from the personal spheres of all others. Suppose we have a set of four alternatives: $A = \{a_1, a_2, a_3, a_4\}$. $\forall D_i(a_1, a_2) : a_1 I_j a_2$ and $\forall D_j(a_3, a_4) : a_3 I_i a_4$.

From here, the social preference will be equivalent to the set of each individual's personal choice set. This observation is true for all pairs. The cases 3 and 4 are to be discussed in the same way: personal choice sets are combined and we obtain the socially proffered alternative, because there can't be any opposition (all individuals are indifferent on the decisivities of all others). The theorem is proved: if each individual has a preference relation which generates a non empty choice set on his *x – aspects* set, then we will have a non-empty social choice set. *q.e.d.*

7. Conclusions

This article aimed had two goals: first, I have created two acceptability criterions for solutions of Sen's theorem; second, I have introduced a new solution for the inconsistency problem. Concerning my first objective, I have formulated the efficacity criterion and the ideological fidelity criterion. The first is a hard to reject condition because it represents a reasonable request that our solutions for Sen's result must work for societies with more than two individuals, with agendas containing any number of alternatives, and rights systems in which individuals have more than one right for each. I've named this condition "*efficacy for real societies*". The second criterion, the ideological fidelity, is easier to be opposed, but there is a good argument which favors it. As long as we want to give formal representation to a libertarian condition, and to introduce it into social choice research domain, we must retain the essence of libertarianism: in libertarian thought, the libertarian principle dominates all other principles or decision rules. In other words, if we retain a notion of rights and we pretend that it is a libertarian notion, then we must make an ethical hierarchy. The consequence of this hierarchy is to consider irrelevant some sorts of externalities. Concerning my second objective, my solution meets both criterions and offers an elegant answer to the impossibility of a paretian libertarian theorem.

Notes

(1) Social Choice Theory research does not have a clear beginning in Arrow's "Social choice and Individual Values". Before Arrow's work we can identify the works of Ramon Lull, Nicolas Cusanus, Jean-Charles de Borda, Marie Jean Antoine Nicolas Caritat de Condorcet, Lewis Carroll. For more details on SCT history, see Urken (1991) and McLean and Hewitt (1994).

(2) The alternatives will be combined with the operators of propositional logic, but I will use quantifiers from second order logic.

(3) Initially (1970a, 1970b), Sen names his theorem as "*the impossibility of a paretian liberal theorem*". In 1976, he renamed his result as "*the impossibility of a paretian libertarian theorem*". The justification of this new name is, in Sen's words, that of giving up to the less clear term of liberalism.

(4) An externality exists, in Holtermann's terms, „*whenever an output of one economic agent appears as an input in the consumption or production vector of another economic agent without compensation being paid by either party, except as a result of government intervention*" (Holtermann, 1972, p. 79).

(5) For a deeper discussion of the externalities problem in the libertarian inconsistency, see Bernholz (1974, 1975), Campbell and Kelly (1997), Saari (2006).

(6) In the classical form, this principle states that a policy is admissible if and only if at least an individual gains something and nobody loose.

(7) d11 is general and works for any number of individuals and simple alternatives.

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