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Characteristics of Criteria for Selecting Investment Projects under Uncertainty

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Abstract. *Within financial theory and practice, there are used five main criteria for selecting investment projects: the net present value (NPV) criterion, the internal rate of return (IRR) criterion, the return term (RT) criterion, the profitability ratio (PR) criterion and the supplementary return (SR) criterion. The assay will emphasize several new properties of said indexes for investment assessment, having as starting point the hypotheses of (approximately) normal repartition of cash-flows generated by an investment project. The obtained results point to the fact that the NPV indexes (the analysis of this criterion was carried out in the article "The NPV Criterion for Valuing Investments under Uncertainty", Daniel Armeanu, Leonard Lache, Economic Computation and Economic Cybernetics Studies and Research no. 4/2009, pp. 133-143), IRR, PR, RT and SR register normal repartitions, therefore simplifying the investment analysis under economic uncertainty, by the capacity of building confidence intervals and assessing probabilities for the inferior limits of said investment assessment indexes.*

Keywords: internal rate of return; cash flow; topical rate; uncertainty; normal repartition; confidence interval; mathematical expectation; square mean deviation of the net present value.

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Within the present article, we intend to emphasize a new property of the selection criteria of the investment projects, namely the IRR, PR, RT and SR value distribution normality of an investment project under the conditions of uncertainty, the probability that said project be accepted by investors and, the confidence intervals for these indexes with different occurrence probabilities. In this regard, we will consider the following scenario for an investment project:

- the investment initial value is I_0 = 1,000,000 EUR;
- the longevity of the investment project is N = 10 years;
- the used topical rate is k = 12%;
- the investment scrap value is null (SV = 0).

The internal rate of return represents, by definition, that value of the topical rate (discounting rate) for which the net present value of the investment project is equal to 0. IRR results as a solution of the equation:

$$\sum_{i=1}^{n} \frac{CFD_i}{(1+k)^i} - I_0 = 0 \qquad (1)$$

The IRR calculation is based on the hypothesis that the future cash flows can be reinvested at this rate of return, yet a less plausible hypothesis. In case of our project, the rated values of available cash flows generated by the project are:

No.	p_i	CFD	CFD_2	CFD_3	CFD_4	CFD_5	CFD_6	CFD_7	CFD_8	CFD_9	CFD_1
1	0.02	46.3	52.95	63.54	74.13	79.42	95.40	100.6	76.78	70.23	52.95
2	0.03	80.2	91.72	110.0	128.4	137.5	160.5	174.2	132.9	123.3	91.72
3	0.04	101.	115.8	138.9	162.1	173.7	202.6	220.0	167.9	149.7	115.8
4	0.05	110.	126.0	151.2	176.4	189.0	220.5	239.4	182.7	161.0	126.0
5	0.06	118.	135.6	162.7	189.8	203.4	237.3	257.7	196.6	172.9	135.6
6	0.07	125.	145.8	172.3	201.0	215.4	251.3	277.7	208.1	183.1	143.6
7	0.09	165.	189.0	226.8	264.6	284.0	330.7	363.1	278.3	241.0	189.0
8	0.15	175.	200.5	239.5	283.4	301.5	353.2	380.1	291.0	255.7	199.9
9	0.12	191.	220.1	266.4	305.9	329.9	383.2	414.9	316.8	278.6	218.5
10	0.1	204.	232.5	279.0	325.5	348.7	406.8	441.7	337.1	296.4	232.5
11	0.09	208.	237.0	284.4	331.8	355.5	414.7	450.3	343.6	302.2	237.0
12	0.07	217.	246.8	296.1	345.5	370.2	431.9	468.9	357.8	314.6	251.3
13	0.06	247.	283.3	340.0	396.7	425.0	495.8	538.3	410.8	361.2	284.1
14	0.04	254.	288.9	348.3	406.4	435.4	508.0	551.6	420.9	370.1	290.3
15	0.01	261.	299.0	358.8	418.6	448.5	523.3	568.2	433.6	381.3	299.0

The IRR rating involves the use of numerical analyses (for instance, Newton-Rhapson procedure). Without entering into details, we present the results obtained for each of the 15^{th} nature conditions:

No.	RIR_i (%)	p_i
1	-5.5582	0.02
2	3.7667	0.03
3	8.336	0.04
4	10.1196	0.05
5	11.7369	0.06
6	13.1279	0.07
7	19.8989	0.09
8	21.4612	0.15
9	23.9674	0.12
10	25.6886	0.1
11	26.2572	0.09
12	27.5321	0.07
13	31.9274	0.06
14	32.7213	0.04
15	33.7700	0.01

The mathematical expectation and risk (square mean deviation) of the IRR variate are evaluated by using the formulas:

$$E(RIR) = \sum_{i=1}^{15} p_i \times RIR_i \quad (2)$$

$$\sigma(RIR) = \sqrt{\sum_{i=1}^{15} p_i \times [RIR_i - E(RIR)]^2} \quad (3)$$

After evaluation, we obtain:

$$E(RIR) = 20.7721\%$$

$$\sigma(RIR) = 8.2429\%$$

We now intend to test the normality of distribution for IRR values. In this sense we will use the well-known Kolmogorov-Smirnov normality test, whose mechanism we will briefly disclose in the followings.

The econometrical Kolmogorov-Smirnov (KS) test is centered on the idea of building an empirical probability distribution based on the data in the sample. This distribution, noted $F_N(x)$, has the following definition:

$$F_N(x) = \frac{1}{N} \sum_{k=1}^{N} I_{X_k \le x} \quad (4)$$

where S_k represents the carried out observations, and represents the indicator[(1)] distribution (or Heaviside function).

Starting form this empirical probability distribution and, from the supposition that the rates or values in the sample come from a normal probability distribution, whose probability distribution is F(x), the KS statistics is thus defined:

$$KS_N = \max_x \left| F_N(x) - F(x) \right| \quad (5)$$

The KS_N statistics-test is 0 concurrent when the carried out observations come from a distribution whose probability distribution is F(x). This also represents the null hypothesis of the KS test. It can be demonstrated that, if the two probability distributions tend to "overlap", then, at limit, the following relation takes place:

$$\lim_{N \to \infty} \sqrt{N} \times KS_N = K \quad (6)$$

where K represents the tabled rate of Kolmogorov probability distribution. The decision of accepting or rejecting the null hypothesis if the KS test is based, as in case of any econometrical test, on the comparison between the statistics rate-test and the critical rates of K.

In our case, the KS_N statistics rate-test equals 0.10287, and K (for a significance threshold of 5%) is $K_{0.95} = 1.3581$. As $KS_N < K$, we accept the null hypothesis of the test, namely the normality of distribution for IRR variate.

As IRR follows a normal distribution, we may establish the probability that the internal rate of return for the investment project exceed the topical rate k (this represents one of the main selection criteria of investment projects):

$$P(RIR > k) = P\left(\frac{RIR - E(RIR)}{\sigma(RIR)} > \frac{k - E(RIR)}{\sigma(RIR)} \right) = 1 - N\left(\frac{k - E(RIR)}{\sigma(RIR)} \right) \quad (7)$$

where N(x) represents the theoretical probability distribution of standard Gaussian distribution N(0,1). On our example, wherein k = 12%, we obtain:

$$P(RIR > 0{,}12) = 1 - N\left(\frac{0.12 - 0.207721}{0.082429} \right) = 0.856381$$

Therefore our project has over 85% chances to succeed. This value is very close to the probability that NPV be positive (P(NPV > 0) = 84.1345%), which represents an empirical reasoning of the fact that, in case of the present project, NPV and IRR criteria are equivalent.

The Table below presents different possible values for the inferior IRR limit, along with the associated probabilities[(2)]:

RIR_1(%)	$P(RIR > RIR_1)$ (%)
5	97.2153
7	95.2618
10	90.4366
12	85.6381
13	82.7130
15	75.8115
16	71.8683
18	63.1678
20	53.7313
21	48.8971
22	44.0790
23	39.3471
25	30.4004
27	22.4959
30	13.1463

Proceeding in the same manner with the confidence interval for NPV (Armeanu, Lache, 2009, pp. 133-143), there can de established a similar interval for the internal rate of return. Thus, for a significance threshold $\alpha = 1 - \delta$, we have:

$$\left[E(RIR) - z_{1-\frac{\alpha}{2}} \times \sigma(RIR), E(RIR) + z_{1-\frac{\alpha}{2}} \times \sigma(RIR) \right]$$

For different δ probabilities there are obtained the following IRR confidence intervals:

Probabilities δ (%)	confidence intervals (%)
60	[13.8347 – 27.7095]
65	[13.0684 – 28.4758]
70	[12.2289 – 29.3153]
75	[11.2899 – 30.2543]
80	[10.2084 – 31.3358]
85	[8.9062 – 32.6380]
90	[7.2138 – 34.3304]
95	[4.6163 – 36.9278]
96	[3.8433 – 37.7009]
97	[2.8843 – 38.6599]
98	[1.5963 – 39.9479]
99	[-0.4602 – 42.0043]
99.5	[-2.3659 – 43.9101]

I. Return term criterion

The Return Term (RT) expresses the number of years for return, by means on annual cash flows, of the sum allotted for carrying out the investment. RT is evaluated taking into consideration the number of years r for which, starting with year r+1, the cumulative value of cash flows generated by the investment exceeds the value of the initial investment. Assuming that the cash flow generated by the investment in year r+1 is evenly cashed along the entire year, the number of days in year r+1 necessary for the return of the investment is evaluated weighting the 360-day of the year with the ratio between the sum of the investment that is to be returned and, the available cash flow of year r+1:

$$TR = \max_{1 \le r \le 10} \left\{ r \mid \sum_{i=1}^{r} CFD_i \le I_0 \right\} + \frac{I_0 - \sum_{i=1}^{r} CFD_i}{CFD_{r+1}} \qquad (8)$$

that is r years and $\frac{I_0 - \sum_{i=1}^{r} CFD_i}{CFD_{r+1}} \times 360$ days.

Within the specialty literature[(3)], there was ascertained the idea according to which the investments having a return term as short as possible, are preferred. Yet, as it was demonstrated by the financial practice, there are numerous cases on investment projects which, although they are characterized by relatively long return terms, generate an NPV which is superior to other projects with a more reduced RT, thus contributing at maximizing the enterprise vale. In such cases, it is recommended the use of the NPV criterion for selecting investments, although, in the end, said selection process is undoubtedly influenced by the company management vision and by the specific objects the management has in view on said business line.

The values obtained for the investment return term, in the 15[th] considered nature conditions, are:

No. i	p_i	TR_i (years)	TR_i (years and days)
1	0.02	0.00	0
2	0.03	7.88	7 years 318 days
3	0.04	6.48	6 years 143 days
4	0.05	6.11	6 years 40 days
5	0.06	5.80	5 years 288 days
6	0.07	5.56	5 years 200 days
7	0.09	4.54	4 years 196 days
8	0.15	4.34	4 years 122 days

9	0.12	4.05	4 years 18 days
10	0.1	3.87	3 years 315 days
11	0.09	3.82	3 years 294 days
12	0.07	3.69	3 years 250 days
13	0.06	3.32	3 years 117 days
14	0.04	3.27	3 years 97 days
15	0.01	3.19	3 years 70 days

In order to evaluate the mathematical expectation and RT standard deviation, the following formulas are used:

$$E(TR) = \sum_{i=1}^{15} p_i \times TR_i$$
$$\sigma(TR) = \sqrt{\sum_{i=1}^{15} p_i \times [TR_i - E(TR)]^2} \quad (9)$$

Applying the formulas (9), we obtain:

$$\begin{cases} E(TR) = 4.34 \text{ years or } 4 \text{ years } 157 \text{ days} \\ \sigma(TR) = 1.24 \text{ years or } 1 \text{ year } 85 \text{ days} \end{cases}$$

We now propose to test the RT distribution normality. In this regard, we will use again the Kolmogorov-Smirnov test, for which we obtain a statistics value-test $KS_N = 0.1896$, whereas the quantila of range 0.95[4] of the Kolmogorov theoretical distribution is $K_{0,95} = 1.3581$. As in case of IRR, we accept the null hypothesis of the test: the RT variable is normally distributed. Therefore, we may speculate all properties resulting therefrom .

The probability that RT be superior to a randomly selected RT_1 equals

$$P(TR > TR_1) = 1 - N\left(\frac{TR_1 - E(TR)}{\sigma(TR)}\right) \quad (10)$$

The confidence intervals $\delta = 1 - \alpha$ are thus determined:

$$\left[E(TR) - z_{1-\frac{\alpha}{2}} \times \sigma(TR), E(TR) + z_{1-\frac{\alpha}{2}} \times \sigma(TR)\right] \quad (11)$$

The table below presents the confidence intervals for the investment return term at different levels of evaluation δ accuracy:

Probability δ (%)	confidence intervals TR (years)
60	[3,39,5,47]
65	[3,28,5,59]
70	[3,15,5,71]
75	[3,01,5,86]
80	[2,85,6,02]
85	[2,66,6,21]
90	[2,40,6,47]
95	[2,01,6,86]
96	[1,90,6,97]
97	[1,75,7,11]
98	[1,56,7,31]
99	[1,25,7,62]
99,5	[0,96,7,90]

II. The profitability ratio criterion

The Profitability Ratio (PR) measures the relative return of an investment project, taking into account the entire lifetime thereof. Practically, PR shows the profit obtained for an invested monetary unit. It is evaluated by using the formula:

No. i	p_i	IP_i (%)
1	0.02	-61.1775
2	0.03	-32.9329
3	0.04	-15.5273
4	0.05	-8.1423
5	0.06	-1.1614
6	0.07	5.0577
7	0.09	38.1171
8	0.15	46.2935
9	0.12	59.8210
10	0.1	69.5200
11	0.09	72.7781
12	0.07	80.1543
13	0.06	106.5070
14	0.04	111.4439
15	0.01	117.9219

$$IP = \frac{VAN}{I_0} \tag{12}$$

The results obtained for the profitability ratio, within the 15^{th} nature conditions, are disclosed in the following table:

The mean value and the square mean deviation of the profitability ratio is evaluated by using the relations below:

$$E(IP) = \sum_{i=1}^{15} p_i \times IP_i$$
$$\sigma(IP) = \sqrt{\sum_{i=1}^{15} p_i \times [IP_i - E(IP)]^2} \quad (13)$$

The obtained results are as follows:

$$\begin{cases} E(IP) = 45.7379\% \\ \sigma(IP) = 40.4737\% \end{cases}$$

As seen before, NPV represents a normally distributed variate, from relation (12) we conclude that PR also has a normal distribution. Thus, we can determine the probability that PR be superior to a PR_1 value:

$$P(IP > IP_1) = 1 - N\left(\frac{IP_1 - E(IP)}{\sigma(IP)}\right) \quad (14)$$

In the particular case when PR_1 = 0%, the relation (14) becomes:

$$P(IP > 0) = 1 - N\left(-\frac{E(IP)}{\sigma(IP)}\right) \quad (15)$$

The obtained results are:

IP_1 (%)	$P(IP > IP_1)$ (%)
0	87.0776
10	81.1380
15	77.6210
20	73.7585
25	69.5808
30	65.1304
40	55.6369
50	45.8067
65	31.7067
80	19.8629
90	13.7065
100	9.0013
110	5.6171
135	1.3712
150	0.4997
180	0.0455

From the table it results that the probability for PR be positive is of about 87%, namely very close to the probabilities that NPV be positive (about 84%) and IRR be superior to the topical rate k (approximative 85%), which demonstrates the equivalence of NPV, IRR and PR criteria for our project.

At the same time, we may determine confidence intervals δ for PR, based on the formula:

$$\left[E(IP) - z_{1-\frac{\alpha}{2}} \times \sigma(IP), E(IP) + z_{1-\frac{\alpha}{2}} \times \sigma(IP) \right] \quad (16)$$

The confidence intervals, for different levels of accuracy δ, are disclosed in the table below:

Probabilitatea δ (%)	Intervalul de încredere pentru IP (%)
60	[11.6744 – 79.8014]
65	[7.9117 – 83.5642]
70	[3.7897 – 87.6862]
75	[-0.8209 – 92.2968]
80	[-6.1312 – 97.6070]
85	[-12.5252 – 104.0010]
90	[-20.8353 – 112.3112]
95	[-33.5890 – 125.0648]
96	[-37.3848 – 128.8607]
97	[-42.0936 – 133.5694]
98	[-48.4179 – 139.8937]
99	[-58.5153 – 149.9912]
99.5	[-67.8730 – 159.3488]

The fact that the confidence intervals are not compact, even for lower levels of δ probability, may be explained by means of the large standard deviation of the profitability ratio.

III. The supplementary return criterion

The supplementary return (SR) represents another measure of the investment return and is evaluated by using the following formula:

$$RS = \sum_{j=1}^{10} \frac{CFD_j - I_0}{I_0} = \sum_{j=1}^{10} \frac{CFD_j}{I_0} - 1 \quad (17)$$

The investment selection criterion is constituted, the way it was expected, by the maximization of the supplementary return. For the 15th nature conditions considered in case of our project, the obtained supplementary returns are:

No. i	p_i	RS_i (%)
1	0.02	-28.7630
2	0.03	23.0886
3	0.04	54.8302
4	0.05	68.3013
5	0.06	81.0779
6	0.07	92.4532
7	0.09	153.2138
8	0.15	168.0181
9	0.12	192.5916
10	0.1	210.5030
11	0.09	216.4936
12	0.07	230.1193
13	0.06	278.3648
14	0.04	287.4402
15	0.01	299.2451

The evaluation of the mathematical expectation and supplementary return volatility is based on the already known formulas:

$$E(RS) = \sum_{i=1}^{15} p_i \times RS_i$$
$$\sigma(RS) = \sqrt{\sum_{i=1}^{15} p_i \times [RS_i - E(RS)]^2} \quad (18)$$

After performing the evaluation, the following results are obtained:

$$\begin{cases} E(RS) = 167\% \\ \sigma(RS) = 74.1201\% \end{cases}$$

As the available cash flows represent normally distributed variates, from the relation (17) also results that the SR variable has a normal distribution. Thus, the probability that SR be superior to an indefinite SR_1 value has the expression:

$$P(RS > RS_1) = 1 - N\left(\frac{RS_1 - E(RS)}{\sigma(RS)}\right) \quad (19)$$

In the particular case wherein $SR_1 = 0$, we have:

$$P(RS > 0) = 1 - N\left(-\frac{E(RS)}{\sigma(RS)}\right) = 98.7874\%$$

The table below discloses the probabilities that SR be superior to SR_1 lower limits, arbitrarily selected:

SR_1 (%)	$P(SR > SR_1)$ (%)
-30	99.6068
-10	99.1530
0	98.7874
20	97.6331
30	96.7724
40	95.6684
50	94.2777
60	92.5575
75	89.2739
90	85.0564
100	81.6986
120	73.6994
140	64.2173
170	48.3857
200	32.8079
220	23.7288

The confidence intervals for SR (for different levels of the α significance threshold) is thus determined:

$$\left[E(RS) - z_{1-\frac{\alpha}{2}} \times \sigma(RS), E(RS) + z_{1-\frac{\alpha}{2}} \times \sigma(RS)\right] \tag{20}$$

The table below contains the confidence intervals for the supplementary return SR:

Probability δ (%)	confidence intervals RS (%)
60	[108.5826 – 225.4174]
65	[108.0512 – 225.9488]
70	[107.5307 – 226.4694]
75	[107.0210 – 226.9790]
80	[106.5225 – 227.4776]
85	[106.0350 – 227.9651]
90	[105.5586 – 228.4415]
95	[105.0934 – 228.9066]
96	[105.0017 – 228.9983]
97	[104.9105 – 229.0896]
98	[104.8197 – 229.1804]
99	[104.7293 – 229.2707]
99.5	[104.6843 – 229.3157]

Grounding the decision of investing within the probabilistic environment is carried out by evaluating the cash-flow-states afferent to each year, based on the variations registered in previous years or by subjectively evaluating the investors and the use thereof in the net present value formula. Given the determinist environment, when cash flows were known for sure, in the non-determinist environment different values of cash flows are assumed as known, on different nature conditions with particular occurrence probabilities. The probabilities of occurrence are quantified either given a history thereof, when such a history exists, or based on the previous frequency, or they are subjectively estimated by specialists. The novelty of the present article consists in the fact that we succeeded to prove the IRR, PR, RT and SR values distribution normality and determine the probability that said project be accepted by investors, as well as the confidence intervals for said ratios with different occurrence probabilities, allowing the investors the rigorous grounding of the decision of investing under uncertainty.

Notes

(1) This indicator function has the following expression: $I_{X_k \le x} = \begin{cases} 1, & X_k \le x \\ 0, & X_k > x \end{cases}$.

(2) The probability evaluation methodology is, $P(RIR > RIR_1) = 1 - N\left(\frac{RIR_1 - E(RIR)}{\sigma(RIR)}\right)$.

(3) See Stancu (2007), and other studies.

(4) We considered the same significance threshold of 5%.

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