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Optimum Repartition of Transport Capacities in the Logistic System using Dynamic Programming

Gheorghe BĂŞANU
Bucharest Academy of Economic Studies
gheorghe.basanu@man.ase.ro
Victor TELEAŞĂ
Bucharest Academy of Economic Studies
victorteleasa@yahoo.com
Eduard ARMEANU
Bucharest Academy of Economic Studies
armeanu@gmail.com

Abstract. *Transportations take an essential role in logistics, interconnecting the majority of processes and operations within logistic system. The efficient use of transportation capacity is a priority whose achievement can diminish logistic costs. This objective is today difficult to achieve due to increasing complexity of transportation monitoring and coordination. This complexity is determined by transportation number and diversity, by the volume and diversity of orders, by increasing the targets to be supplied.*

Dynamic programming represents a highly useful tool for logistic managers, considering that its specific techniques and methods are oriented toward solving problems related to resource optimum allocation and utilization.

The present paper presents briefly a series of theoretical elements of dynamic programming applied in logistics, based on which it is shown a mathematic model to determine the optimum policy for transport capacity repartition for the area attached to a logistic centre, through three distribution centres.

Keywords: distribution centre; logistics; number of customers; stage; dynamic programming.

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A logistic system has as main objective to ensure the circuit of material and product flows to users and consumers. This objective can only be reached in certain conditions, required both by customer demands and by the necessity to efficiently use the available logistic capacities.

The resources required for performing transport operations within logistic system are set based on a detailed analysis of all elements on which the delivery of materials and products to customers, in quantity and of quality they request, is based. Identifying and determining which customers shall be supplied, the transport routes, the characteristics and quantities of materials and products to be supplied, the number of freight carriers and the personnel requested for performing the corresponding logistic operations is a laborious assessment process on which the final decision, oriented toward efficient use of available resources is based.

Within a logistic system, the supplying of consumption centres is made, as usually, through distribution centres equipped with road carriers of various types and capacities. The demand for materials and products varies extremely in time and this is why the main problem of logistic managers is meeting in a more efficient and flexible manner, these demands, by using a lower number of freight carriers. In this context, the efficiency means using the full available transport capacity at the same time with diminishing the distances for freight carriers. This way one sets a balance between the degree of conformity for logistic services and the customer demand and suppliers' costs required for performing these services.

From the flexibility point of view, the transportation within logistics must meet two basic requirements: to ensure, in due time, the capacity required for transporting the entire quantity of products and materials requested by customers, and to ensure the compatibility between the freight carriers which are used and the specific characteristics of materials and products. These requirements are obviously interconnected and only together they guarantee completely, as quantity and quality, the orders of customers.

The level of stocks of materials and products can be drastically diminished, by creating a continuous flow of raw materials and materials within logistic system, in which the transport synchronization has an essential role and is reflected in the rhythm of supply of production lines. The logistic costs can be drastically diminished, by ensuring a rational dispersion of transport capacities within the supplied logistic area. This way, the deliveries of materials and products in small quantities and with high frequency can be performed with low capacity freight carriers, while the deliveries of materials and products in high quantities, to various supplying centres, can be performed in circuit using freight carriers of medium and high capacity.

The dynamic programming is a method for solving the optimization problems which requires that the best policy to be adopted by determining, for each decision composing such a policy, sub-problems to be solved, so that to find an optimum solution for the initial problem, among the optimum solutions of these sub-problems. This thesis is based on the *optimality principle* (Stokey, Lucas, 1989) according to which (Bellman, 1957) *an optimal policy has the propriety that whichever would be the original decision ad status, the following decisions must represent an optimal policy regarding the status resulted from the original decision*. As a consequence, *any optimal policy can only be composed of other optimal sub-policies* (Kaufmann, 1967). This principle shows us that, if a policy contains a sub-policy which is not optimal, then its replacement with an optimal one shall determine the improvement of the original policy.

It worth mentioning that dynamic programming is not offering us a standard mathematic formula for solving all problems. Depending on the available variables and on the set objective, for each case one must identify the corresponding recurrence equation able to lead to the optimal solution. The dynamic programming problems can be deterministic (as the one treated in this paper) or probabilistic, as the result of each decisional step is unique and known, respectively it is a probability distribution.

Having available a resource quantity *r* for performing *n* activities, the problem is how to distribute these resources to obtain an optimal result. The mathematic formula of such a general case would be:

$$\text{optimization: } z = \sum_{j=1}^{n} g_j\left(x_j\right), \tag{1}$$

in the following conditions:

$$\sum_{j=1}^{n} a_j x_j \le r \text{, with all variables being non-negative whole} \tag{2}$$

where:

$g_1\left(x_1\right), g_2\left(x_2\right), \ldots, g_n\left(x_n\right)$ - utility functions;

$x_1, x_2, \ldots, x_n$ - decisional variables;

r and *n* – whole and non-negative numbers.

We assume that the optimization in our case aims *the maximization* of the function *z*, and the maximal result obtained subsequently to optimal allocation of *r* quantity of resources for performing the *n* activities is represented by the function $f_n\left(r\right)$. As a result, we have:

$$f_n(r) = \max z . \qquad (3)$$

In case for n-1 activities we optimally allocate the quantity r_1 of resources, we can sate that the optimal result obtained shall be defined by the function $f_{n-1}(r_1)$. By assigning a quantity x_n of resources to the activity n, with $0 \le x_n \le r$, we deduce that, for n-1 activities left, we shall have available the following quantity of resources:

$$r_1 = r - x_n . \qquad (4)$$

The relation (1) becomes:

$$z = g_n(x_n) + f_{n-1}(r_1) = g_n(x_n) + f_{n-1}(r - x_n) . \qquad (5)$$

Considering the relation (3), we obtain the recurrence relation (Bellman, 1957, Stokey, Lucas, 1989):

$$f_n(r) = \max_{0 \le x_n \le r} \left[g_n(x_n) + f_{n-1}(r - x_n) \right], \qquad (6)$$

for $n = 2,3,\ldots$ and $r \ge 0$, and $f_1(r) = g_1(r)$.

The relation (6) represents *the fundamental equation* of dynamic programming, which ensures the above mentioned principle of optimality.

For dynamic programming problem solving first one sets the utility functions, the decisional variables, the aimed function, the restrictions and steps to follow, after which one determines the equation of recurrence by which, using successive iterations, one obtains the optimal policy.

In order to increase the efficiency of a logistic centre, one decided to create three types of distribution centres which shall have their own motor vehicle fleet and which shall supply a determined logistic area, defined by a number of supplying centres (customers). The main characteristics that particularize one distribution centre from another are: the number and type of freight carrier with which they shall be equipped (depending on the nature of stored materials and products and by the dimensions of future loading unit), and the number of customers to be supplied within the competent area of the logistic centre. The amount assigned for investments allows the purchase of maximum 42 motor vehicles of various capacities. The logistic managers must identify how many distribution centres of each type can create so that the number of supplied customers (working outlets) to be maximum, considering the data within Table 1.

Table 1

The characteristics of distribution centre types

Distribution centre type	Number of motor vehicles within fleet	Number of supplied customers (α_i)	Motor vehicle capacity
A	15	35	Low
B	20	68	Medium
C	12	47	High

Based on the data within Table 1, the mathematic model includes the relations between the utility functions, respectively resource assignment function,

$$\begin{cases} g_1(x_1) = 35x_1 \\ g_2(x_2) = 68x_2 \\ g_3(x_3) = 47x_3 \end{cases} \quad (7) \qquad \begin{cases} h_1(x_1) = 15x_1 \\ h_2(x_2) = 20x_2 \\ h_3(x_3) = 12x_3 \end{cases} \quad (8)$$

aimed function,

$$\max z = 35x_1 + 68x_2 + 47x_3 \quad (9)$$

and restrictions:

$$15x_1 + 20x_2 + 12x_3 \le 42 \quad (10)$$

$$x_j \ge 0, (j = 1,2,3)$$

x_j whole numers.

The first restriction sets the maximum number of motor vehicles which can be assigned to all distribution centres, and the second and third restrictions set that the number of created distribution centres must be positive and whole.

Considering that the problem aims to create three type of distribution centres, it results that three steps must be followed, each of them having more statuses, determined by the number of assigned motor vehicles, so that the number of supplied customers to be maximum.

Therefore, during the first step, based on the fundamental equation given by the relation (6), the function defining the maximum number of customers supplied by the A type distribution centre, when assigning an appropriate number of transportation vehicles, shall be:

$$f_A(r_1) = \max_{x_1}[35x_1]. \quad (11)$$

Designing by ξ_i the values of the decisional variables x_j, ($\xi_i \ge 0, \xi_i$ whole), which is in fact the number of distribution centres of each type which shall be created, depending on the available number of motor vehicles for the A type distribution, considering the expression of the function $h_1(x_1)$, we obtain the relation:

$$f_A(r_1) = 15\xi_1 . \tag{12}$$

Which must comply with the condition: $15\xi_1 \leq r_1$ (13)
where r_1 represents the maximum number of motor vehicles which can be assigned the A type distribution centre, till unknown during this step. As the values of the decisional variables must be positive and whole, the number of the A type distribution centres shall be given by the relation:

$$\xi_1 \leq [r_1/15] \tag{14}$$

of which one shall only consider the obtained whole value immediately inferior.

This way the relation (11) becomes:

$$f_A(r_1) = 35\xi_1 \tag{15}$$

which represents the total number of customers supplied by the created A type distribution centres.

During the second step, by applying the same relation (6) we obtain:

$$f_{AB}(r_2) = \max_{x_2}\left[g_2(x_2) + f_A(r_1)\right] = \max_{x_2}\left[68x_2 + f_A(r_1)\right]. \tag{16}$$

Considering r_2 as the maximum number of motor vehicles assigned to A type and B type distribution centres, until this step and the expression of the function $h_2(x_2)$, we can rewrite the relation (4), for this step as follows:

$$r_1 = r_2 - 20\xi_2 . \tag{17}$$

By replacing the relation (17) within the relation (16) we obtain:

$$f_{AB}(r_2) = \max_{\xi_2}\left[68\xi_2 + f_A(r_2 - 20\xi_2)\right]. \tag{18}$$

During the third step, we consider r_3 as the maximum number of motor vehicles to be assigned, for optimization, to the three distribution centres. Considering that it is the last utility function, it results that it is represented by the maximum number of motor vehicles which can be purchased$(r_3 = 42)$. Together with the expression of the function $h_3(x_3)$ we can rewrite the relation (4), for this step as follows:

$$r_2 = r_3 - 12\xi_3 = 42 - 12\xi_3 . \tag{19}$$

During this step the relation (6) shall be:

$$f_{ABC}(r_3) = \max_{x_3}\left[g_3(x_3) + f_{AB}(r_2)\right] = \max_{x_3}\left[47x_3 + f_{AB}(r_2)\right]. \tag{20}$$

By replacing the relation (19) within the relation (20) we obtain the relation:

$$f_{ABC}(r_3) = \max_{\xi_3}\left[47\xi_3 + f_{AB}(r_3 - 12\xi_3)\right]. \tag{21}$$

The first step of the iterative determination algorithm for the number of customers which can be supplied by the A type distribution centre is defined by the relations (11) ÷ (15), which together with the data within Table 1 represents the starting point for the calculations within next steps.

During the second step, we designate by x_2^* the maximum number of customers which can be supplied by the created A type and B type distribution centres, the values of calculation variables and the obtained results, after following the iterations within this step, are shown in Table 2.

Table 2

Maximum number of customers which can be supplied by created A type and B type distribution centres, depending on the assigned number of motor vehicles

Iteration no	ξ_3	ξ_2	r_2	r_1	ξ_1	$f_A(r_1)$	x_2^*
1	0	0	42	42	2	70	70
2	0	1	42	22	1	35	103
3	0	2	42	2	0	0	136
4	1	0	30	30	2	70	70
5	1	1	30	10	0	0	68
6	2	0	18	18	1	35	35
7	3	0	6	6	0	0	0

Using the significant of the notations used so far, within a first set of iterations, maintaining $\xi_3 = 0$, we successively assign various values to ξ_2, starting from 0.

- For $\xi_3 = 0$, $\xi_2 = 0$ and $r_2 = 42$ (determined by the relation (19), for $\xi_3 = 0$), by applying the relation (17) we obtain:

$$r_1 = r_2 - 20\xi_2 = 42 - 20 \cdot 0 = 42,$$

and using the condition (14) we obtain:

$\xi_1 = [r_1/15] = [42/15] = 2$.

By replacing this value in the relation (15) we obtain:

$f_A(r_1) = 35\xi_1 = 35 \cdot 2 = 70$

the relation (18) becoming,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 0 + 70 = 70$

The results obtained are shown in Table 2 to iteration 1.

• For $\xi_3 = 0$, $\xi_2 = 1$ and $r_2 = 42$, we obtain:

$r_1 = r_2 - 20\xi_2 = 42 - 20 \cdot 1 = 22$

$\xi_1 = [r_1/15] = [22/15] = 1$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 1 = 35$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 1 + 35 = 103$

The results obtained are shown in Table to iteration 2.

• For $\xi_3 = 0$, $\xi_2 = 2$ and $r_2 = 42$, we obtain:

$r_1 = r_2 - 20\xi_2 = 42 - 20 \cdot 2 = 2$

$\xi_1 = [r_1/15] = [2/15] = 0$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 0 = 0$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 2 + 0 = 136$

The results obtained are shown in Table 2 to iteration 3.

• For $\xi_3 = 0$, $\xi_2 = 3$ and $r_2 = 42$, we obtain $r_1 = r_2 - 20\xi_2 = 42 - 20 \cdot 3 = -18$. Considering that we have a negative value, this iteration shall be cancelled.

Hereinafter, we shall initiate a second set of iterations, maintain constant $\xi_3 = 1$ and assigning successively values for ξ_2, starting from 0.

• For $\xi_3 = 1$, $\xi_2 = 0$ and $r_2 = 30$ (determined by the relation (19), for $\xi_3 = 1$), we obtain:

$r_1 = r_2 - 20\xi_2 = 30 - 20 \cdot 0 = 30$

$\xi_1 = [r_1/15] = [30/15] = 2$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 2 = 70$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 0 + 70 = 70$

He results obtained are shown in Table 2 to iteration 4.

• For $\xi_3 = 1$, $\xi_2 = 1$ and $r_2 = 30$, we obtain:

$r_1 = r_2 - 20\xi_2 = 30 - 20 \cdot 1 = 10$

$\xi_1 = [r_1/15] = [10/15] = 0$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 0 = 0$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 1 + 0 = 68$

The results obtained are shown in Table 2 to iteration 5.

• For $\xi_3 = 1$, $\xi_2 = 2$ and $r_2 = 30$, we obtain:

$r_1 = r_2 - 20\xi_2 = 30 - 20 \cdot 2 = -10$. Considering that we have a negative value, this iteration shall be cancelled.

N the third set of iterations, we maintain constant $\xi_3 = 2$ and we successively assign various values for ξ_2, starting from 0.

• For $\xi_3 = 2$, $\xi_2 = 0$ and $r_2 = 18$ (determined by the relation (19), for $\xi_3 = 2$), we obtain:

$r_1 = r_2 - 20\xi_2 = 18 - 20 \cdot 0 = 18$

$\xi_1 = [r_1/15] = [18/15] = 1$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 1 = 35$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 0 + 35 = 35$

The results obtained are shown in Table 2 to iteration 6.

• For $\xi_3 = 2$, $\xi_2 = 1$ şi $r_2 = 18$, we obtain:

$r_1 = r_2 - 20\xi_2 = 18 - 20 \cdot 1 = -2$. Considering that we have a negative value, this iteration shall be cancelled.

In the fourth set of iterations, we maintain constant $\xi_3 = 3$ and we successively assign various values for ξ_2, starting from 0.

• For $\xi_3 = 3$, $\xi_2 = 0$ and $r_2 = 6$ (determined by the relation (19), for $\xi_3 = 3$), we obtain:

$r_1 = r_2 - 20\xi_2 = 6 - 20 \cdot 0 = 6$

$\xi_1 = [r_1/15] = [6/15] = 0$, and $f_A(r_1) = 35\xi_1 = 35 \cdot 0 = 0$, resulting,

$x_2^* = f_{AB}(r_2) = 68 \cdot \xi_2 + f_A(r_1) = 68 \cdot 0 + 0 = 0$

The results obtained are shown in Table 2 to iteration 7.

After analysing the data within Table 2 it notes that the best logistic policy is $p^* = (0,2,0)$, respectively the one resulted from the iteration no 3, for which the number of supplied customers is maximum, respectively 136.A such a logistic policy aims to create 2 B type distribution centres, equipped with a total number of 40 medium capacity vehicles.

During the third step, we try to improve this policy by initiating a new analysis, from the perspective of the maximum number of vehicles which can be assigned to each distribution centre, in order to obtain the maximum number of supplied customers during this step, x_3^*. This analysis shall contain a new range of values for ξ_3.

• First, we consider that we shall not create any C type distribution centre. Using the relation (19), for $\xi_3 = 0$, we obtain the number of vehicles available for being assigned to A type and B type distribution centres, to be created:

$$r_2 = r_3 - 12\xi_3 = 42 - 12\xi_3 = 42 - 12 \cdot 0 = 42$$

The relation (18) becomes in this case:

$f_{AB}(r_2) = \max f_{AB}(42) = 136$, resulted obtained by the iterations 1÷3, within the previous step. By applying the relation (21), we obtain:

$x_3^* = f_{ABC}(r_3) = 47 \cdot \xi_3 + f_{AB}(r_2) = 47 \cdot 0 + 136 = 136$, respectively the maximum number of customers supplied by the maximum number of customers supplied by the 2 B type distribution centres created so far (in Table 2, to this value corresponds $\xi_2 = 2$).

• We consider that we shall create one C type distribution centre. By applying the same relation (19), for $\xi_3 = 1$, we obtain the number of available vehicles to be assigned to created distribution centres :

$$r_2 = r_3 - 12\xi_3 = 42 - 12\xi_3 = 42 - 12 \cdot 1 = 30$$

The relation (18) becomes in this case:

$f_{AB}(r_2) = \max f_{AB}(30) = 70$, result obtained after 4 and 5 iterations, from the previous step. By applying the relation (21), we obtain:

$x_3^* = f_{ABC}(r_3) = 47 \cdot \xi_3 + f_{AB}(r_2) = 47 \cdot 1 + 70 = 117$, respectively the maximum number of customers supplied by all the distribution centres created so far. Knowing that we started this iteration by creating a C type distribution centre C, which according to Table 1 shall supply 47 customers, we still have left (117 – 47) = 70 customers to supply. As the number of distribution centres must be whole and positive, also from Table 1 we deduce that we still can create (70:35) = 2 A type distribution centres.

Acting similarly, hereinafter we obtain,

• For $\xi_3 = 2$:

$r_2 = r_3 - 12\xi_3 = 42 - 12\xi_3 = 42 - 12 \cdot 2 = 18$

$f_{AB}(r_2) = \max f_{AB}(18) = 35$

$x_3^* = f_{ABC}(r_3) = 47 \cdot \xi_3 + f_{AB}(r_2) = 47 \cdot 2 + 35 = 129$, which represents the maximum number of customers supplied by all distribution centres. Knowing that we started the iteration by creating 2 C type distribution centres, and having left a number of (129 – 94) = 35 customers to be supplied, we deduce that, following the above conditions, we can create one more A type distribution centre.

• For $\xi_3 = 3$:

$r_2 = r_3 - 12\xi_3 = 42 - 12\xi_3 = 42 - 12 \cdot 3 = 6$

$f_{AB}(r_2) = \max f_{AB}(6) = 0$

$x_3^* = f_{ABC}(r_3) = 47 \cdot \xi_3 + f_{AB}(r_2) = 47 \cdot 3 + 0 = 141$, which represents the number of customers supplied by a number of 3 C type distribution centres.

The results obtained during this step are shown in Table 3.

Analyzing the data within Tables 2 and 3, we can prioritize the optimal policies of transportation capacity repartition by centre of distribution (CD), as shown in Table 4.

Table 3

Maximum number of customers to be supplied by the three created distribution centres, depending on the number of assigned vehicles

Iteration no.	ξ_3	r_2	$f_{AB}(r_2)$	x_3^*	Order of policy
1	0	42	136	136	(2)
2	1	30	70	117	(4)
3	2	18	35	129	(3)
4	3	6	0	141	(1)

Table 4

Hierarchy of logistic policies

Policy	Number of DCs to be created, by type:			Maximum number of customers to be supplied by created DCs	Number of vehicles required for CDC equipment
	A type	B type	C tpe		
1	**-**	**-**	**3**	**141**	**36**
2	-	2	-	136	40
3	1	-	2	129	39
4	2	-	1	117	42
5	1	1	-	103	35

Using the data within Table 4 we drew the chart within Figure 1.

It easily notes that the optimal logistic policy to implement under the given circumstances is $p_{opt} = p_1 = (0,0,3)$, respectively to create 3 C type distribution centres, which shall supply a number of 141 customers and whose equipment requires 36 road vehicles, of those 42 to be potentially purchased.

The efficient use of the available transport capacity represents an essential objective toward increasing the performances of logistic systems. Such an objective can be reached by identifying and implementing methods and models within operational research area. The purpose of such an action shall be reflected in the mathematic formula by the tendency to increase the value of the efficiency criterion which shall completely replace the purpose of the operation (Ghermeier, 1973). The dynamic programming represents one of the methods of system optimization, in which the problem solving mechanism treats their discomposure into steps, the optimizations within each step having reclusive character.

Regardless if the optimal value of the decisions are determined by table calculations or by analytical methods, their successive determination proved to be much more efficient. In order to be able to implement the methods of dynamic programming it is necessary that the assessments (of costs) Associated to decisions be additive, and the path through which the system reached a certain status must not influence the future statuses (Ackoff, Sasieni, 1975).

In order to increase the performance of logistic system and to diminish the costs, from the transportation point of view, it is necessary to:

- fully use the available capacities;
- use on a large scale the IT software for setting the transportation routes, for monitoring and coordinating the specific operations;
- drastically diminish or to eliminate empty carrier routes;
- equip the distribution centres with minimum required freight carrier number;

- maximize the lifetime of freight carriers;
- use on a large scale the pallets and containers;
- use appropriate freight carriers for material and product characteristics;
- fully and duly meet customer requierements.

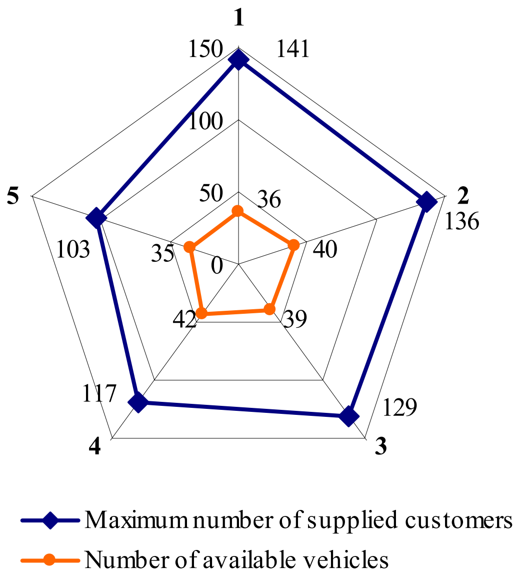

Figure 1. *Logistic policies' chart*

Obtaining an increased transportation capacity involves two issues – an investing one and an organizing one. The first issue considers the strategic side and treats the acquisition of new freight carriers, in accordance with the developing perspectives of logistic system. The second issue treats two sides – tactical and operational – in which the tactical side refers to using additional capacities, on limited term, supplied by specialized transporting entities, and the operational side refers to current measures to be taken by logistic managers for purpose of increasing the degree of use for the existing transport capacities.

To fully know and understand the factors influencing the volume, rhythm, quality and efficiency of transport operations within logistics, to identify and to use the modern method and technologies able to ensure prompt deliveries to customers, to increase the freight carriers' degree of occupancy and to maximize the diminishing of costs, represent the main lines of management action for logistic system's functionality and structure improvement.

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