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Specific patterns in portfolio analysis

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Abstract. *In the mid-twentieth century, under an unprecedented growth of the business of trading in securities, the need to provide a modern framework for assessing the performance of portfolios of financial instruments was felt. To that effect, it is noted that over this period, more and more economists have attempted to develop statistical-mathematical models that ensure the evaluation of profitability and portfolio risk securities. These models are considered to be part of "the modern portfolio theory".*

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REL Classification: 11B.

1. The Markowitz model

A. General aspects regarding the Markowitz portfolio of the management model

The first attempts to develop a modern model designed for the evaluation of the portfolio performance tools belongs to the American professor, Harry Markowitz, who proved in his study, "Portfolio Selection. Efficient Diversification of Investments", that the selection of the portfolio can be achieved through the study of “the rate of expected return and portfolio variance or standard deviation as a measure of risk." This is true with certainty.

The model developed by Markowitz is based on a number of assumptions which may be summarized as follows:

- The investors consider each investment alternative as being represented by the distribution of hoped profit probability for a time period;
- The investors maximize the expected utilities within a period of time, while the utility curve maximises the marginal utility of their welfare;
- The investors estimate the risk based on the modifications in the hoped profits;
- The investors take decisions only based on risk and hoped profit, therefore the utility curve is expressed as a function of expected profit and variance of profit;
- For a given level of risk, investors prefer a bigger profit; for a given level of expected profit, investors prefer lesser risk.

The practical use of the Markowitz model allows the determination of the level of individual dispersions of return in financial instruments for both a simplified portfolio of instruments (consisting of only two financial titles), as well as a portfolio composed of "n" financial instruments.

Even if it is about two or more titles on different markets, the construction of an efficient portfolio involves to gradually follow the steps:

- identifying the risk profile-win for each combination of alternative securities in the portfolio;
- predicting what combination of risky titles with minimal variance depending on the aversion degree of each investor;
- determining the complete portfolio by combining the minimum variance portfolio with the risk-free securities that the investor intends to introduce in its portfolio.

B. The profitability and the risk of a portfolio composed of two financial titles

The simplest portfolio that can be analyzed using the model developed by Markowitz is composed of two financial instruments. Thus, we consider that a capital investor has the possibility of choosing to invest their savings in one of the two available financial titles T1 and T2, or he equally has the possibility of creating a portfolio P, dividing the amount that he wants to invest in the two mentioned above titles.

From a mathematical perspective, the investors anticipations about the behavior of the two titles in the future period can be summarized as follows:

$$T_1 \begin{cases} E_1 \\ \sigma_1 \end{cases} \; T_2 \begin{cases} E_2 \\ \sigma_2 \end{cases}, \mathrm{cov}_{12} = \rho_{12} \times \sigma_1 \times \sigma_2$$

where:
E_i - the mathematical expected value and the rate of return of title i;
σ_i - the standard deviation of the rate of return of title i;
ρ_{ij} - the correlation coefficient between the rates of return of titles i and j;
Cov_{ij} - the covariance between the rates of return of titles i and j.

A capital investor has the possibility to form a portfolio combining the two financial titles in the proportion X1 and X2. In this case, the total available amount is invested in T1 (the amount of the first type of financial instrument acquisition), respectively T2 (the amount of the purchase of the second type of financial instrument). In this case we can establish the following relationship for the calculation:

$X_1 + X_2 = 1$with $X_1, X_2 \geq 0$ or $0 \leq X_1 \leq 1$ and $0 \leq X_2 \leq 1$

Given the above, one can determine the mathematical expectation of the rate of return of the portfolio P (Ep), using this formula:

$E_p = X_1 \times E_1 + X_2 \times E_2$

As it can be seen from the above relationship, the yields hope is the weighted average of the bond yields hope, the proportions being the share.

The second element that should be studied in order to characterize the efficiency of the considered portfolio is the scattering rate of the return portfolio P (Vp), which is actually a measure of the risk of the portfolio investment. For this purpose we will use the following mathematical relationship of calculation:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times \mathrm{cov}_{12}$$

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times \rho_{12} \times \sigma_1 \times \sigma_2$$

From the formulas mentioned above, it results that the portfolios variance is significantly influenced by the following elements:

- dispersion of each title included in the portfolio;
- the proportions combining the two financial titles;
- the covariance between the two considered titles.

To complete the analysis based on the above relations, in specialized literature it is recommended to study the correlation between the two financial titles included in the portfolio. Thus, we can see that depending on the coefficient value of correlation between the two financial titles T1 and T2, there can be identified three different cases, which can be summarized as follows:

- **The correlation coefficient is 1 ($\rho_{12} = 1$)**

In this case, we can say that the financial instruments T1 and T2 are perfectly and positively correlated, which means the anticipation to return these securities movements while perfectly consistent, but with different amplitudes. In this situation, it is considered that the portfolio risk is highest, because the factors that influence the two titles are similar and almost equal in intensity action. It also notes that, in this case, changing the share structure of the portfolio securities does not significaly improve the risk level associated with it.

For this correlation value of the degree, the relations on which we can make an assessment of the portfolio, consisting of two financial titles, may be given as follows:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times cov_{12}$$

to write:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times \rho_{12} \times \sigma_1 \times \sigma_2$$

with: ρ_{12}=1

that is:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times \sigma_1 \times \sigma_2 = (X_1 \times \sigma_1 + X_2 \times \sigma_2)^2$$

$$\sigma_p = X_1 \times \sigma_1 + X_2 \times \sigma_2$$

In this case, it is noted that the standard deviation of the the portfolio is equal to the average standard deviation of the financial titles in it. Bringing together the two equations and reporting the performance and portfolio risk P,

$$E_p = X_1 \times E_1 + X_2 \times E_2$$

and:

$$\sigma_p = X_1 \times \sigma_1 + X_2 \times \sigma_2$$

we get the equation:

$$E_p = f(\sigma_p)$$

as space of combining the titles T_1 şi T_2 in plan E – σ.

is known that:

$X_1 + X_2 = 1$

respectively:

$X_2 = 1 – X_1$

In these circumstances, the equation that can determine the mathematical expected value of the portfolios rate of the return $P(E_p)$ is:

$$E_p = X_1 \times E_1 + (1 - X_1)E_2$$

In this case, the share of title T1 in the portfolio P can be determined using the formula:

$$X_1 = \frac{E_p - E_2}{E_1 - E_2}$$

It is also found that if the mathematical expectations of the rate of return of the two titles are not equal ($E_1 \neq E_2$), the standard deviation of the performance of the portfolio can be calculated as follows:

$$\sigma_p = E_p\left(\frac{\sigma_1 - \sigma_2}{E_1 - E_2}\right) + \frac{E_1 \times \sigma_2 - E_2 \times \sigma_1}{E_1 - E_2}$$

- **The correlation coefficient is -1 (ρ_{12} = -1)**

If the correlation coefficient is ρ12 = -1, then T1 and T2 units are perfectly negatively correlated. In such a situation yield expectations on these securities fluctuates perfectly opposites.

It is worth noting that if the two titles are strictly negative correlated it can be reached, in a certain combination, at the total elimination of risk for the portfolio of securities.

Also in this case, the calculation relations of the standard deviation may be transformed as follows:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times cov_{12}$$

To write:

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_2^2 \times \sigma_2^2 - 2X_1 \times X_2 \times \sigma_1 \times \sigma_2$$

That is:

$$\sigma_p^2 = (X_1 \times \sigma_1 - X_2 \times \sigma_2)^2$$

The standard deviation is always positive, so the discution for the expressions sign $(X_1 \times \sigma_1 - X_2 \times \sigma_2)$ that can vary by X_1 and X_2.

For:

$$X_1 > \frac{\sigma_2}{\sigma_1+\sigma_2}$$

This relation, along with the relation $E_p = X_1 \times E_1 + X_2 \times E_2$, allows the determination of the linked equation between E_p şi σ_p.

We obtain:

$$\sigma_p = E_p \frac{\sigma_1 + \sigma_2}{E_1 - E_2} - \frac{E_2 \times \sigma_1 + E_1 \times \sigma_2}{E_1 - E_2}$$

It is a linear relationship shown graphically by a straight line. Part of this line, corresponding to:

$$X_1 > \frac{\sigma_2}{\sigma_1+\sigma_2}$$

It is the place of the securities portfolios obtained from T_1 and T_2.

For

$$X_1 < \frac{\sigma_2}{\sigma_1+\sigma_2}$$

We have:

$$X_1 \times \sigma_1 - X_2 \times \sigma_2 < 0$$

and

$$\sigma_p = -(X_1 \times \sigma_1 - X_2 \times \sigma_2)$$

In a similar manner, if the value of the correlation degree is equal to 1, we obtain the linear equation linking E_p and σ_p.

$$\sigma_p = -E_p \frac{\sigma_1 + \sigma_2}{E_1 - E_2} + \frac{E_2 \times \sigma_1 + E_1 \times \sigma_2}{E_1 - E_2}$$

A part of this straight line, the one corresponding to

$$X_1 < \frac{\sigma_2}{\sigma_1+\sigma_2}$$

Is the law obtained by combining T_1 and T_2 portfolios.

Finally, for

$$X_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

we have $\sigma_p = 0$.

This result must be mentioned separately because it shows that starting from two risky titles is likely that, choosing rigorous the proportions ($0 \leq X_1$ and $X_2 \leq 1$), to construct an un-risky portfolio. This result is possible if T1 and T2 of the securities are perfectly and negative correlated.

- **The correlation coefficient different from +-1 ($\rho_{12} \neq$ +-1)**

If $-1 < \rho_{12} < +1$ (including $\rho_{12} = 0$) the fluctuations anticipated for the securities T_1 and T_2 are not perfectly dependent (positive and negative). It is the general case, where there is a certain degree of correlation between the yield rate due to the fact that all titles follow more or less the fluctuations of the national economy.

A low correlation may lead to significant improvement in the value of the risky portfolio investment. Also, a null value of this coefficient is considered to be a potential source of a 50% reduction of the portfolios risk analysis.

In the general case (the correlation coefficient different from +1 and -1), for a second portfolio of risky titles is obtained the following expression:

$$\sigma_p^2 = X_1^2 \times V_1 + X_2^2 \times V_2 + 2X_1 \times X_2 \times \mathrm{cov}_{12}$$

$$\mathrm{cu} - 1 < \rho_{12} < +1$$

meaning:

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_2^2 \times \sigma_2^2 + 2X_1 \times X_2 \times \sigma_1 \times \sigma_2 \times \rho_{12}$$

As it can be seen, in this case, unlike the previous situations, the term of risk can be reduced as a perfectly square, which makes it more difficult to determine its practical value.

Starting from this equation and from that of E_p ($E_p = X_1 \times E_1 + X_2 \times E_2$), we establish the relation that links the E_p and σ_p.

From the equation of E_p we obtain $X_1 = (E_p-E_2)/(E_1-E_2)$ value that you enter in the equation of th V_p. Developing we get:

$$\sigma_p^2 = E_p^2 \left[\frac{V_1 + V_2 - 2 \times \mathrm{cov}_{12}}{(E_1 - E_2)^2}\right] + 2E_p \frac{E_1(\mathrm{cov}_{12} - V_2) + E_2(\mathrm{cov}_{12} - V_1)}{(E_1 - E_2)^2} + \\ + \frac{E_2^2 \times V_1 + E_1^2 \times V_2 - 2 \times E_1 \times E_2 \times \mathrm{cov}_{12}}{(E_1 - E_2)^2}$$

E-V equation is obtained in terms of a parabola. In E-σ plan, the equation

$$\sigma_p = \sqrt{V_p} = f(E_p)$$

is a hyperbole of which retain a branch, namely that corresponding positive σ_p values.

- ***The analysis of the contribution of a security risk and yields portfolio that is included (case of two securities portfolio)***

An interesting aspect in the analysis of any portfolio of financial instruments is the risk assessment of the contribution of each title and the overall yield of the portfolio to which it belongs.

To illustrate the methodology for analyzing this contribution we consider the case of a portfolio formed from two titles combined T_1 and T_2 ratios X_1 and X_2.

The contribution of each title to the portfolio return is easily expressed and represents the contribution to the yields formation of each title. The sum of all contributions will be exactly this yield.

It is known that $E_p = X_1 \times E_1 + X_2 \times E_2$.

Title 1 is the contribution of X_1E_1 to the portfolios expectancy. This contribution is based on the expectation of return of title and the proportion invested in the title.

In terms of risk, the problem is more complex.

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_2^2 \times \sigma_2^2 + 2X_1 \times X_2 \times \mathrm{cov}_{12}$$

We can write:

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_1 \times X_2 \times \mathrm{cov}_{12} + X_2^2 \times \sigma_2^2 + 2X_1 \times X_2 \times \mathrm{cov}_{12}$$

$$\sigma_p^2 = X_1(X_1 \times \sigma_1^2 + X_2 \times \mathrm{cov}_{12}) + X_2(X_2 \times \sigma_2^2 + X_1 \times \mathrm{cov}_{12})$$

$$\sigma_p^2 = X_1(X_1 \times \mathrm{cov}_{11} + X_2 \times \mathrm{cov}_{12}) + X_2(X_2 \times \mathrm{cov}_{22} + X_1 \times \mathrm{cov}_{12})$$

$$\sigma_p^2 = X_1\mathrm{cov}(T_1, X_1 \times T_1 + X_2 \times T_2) + X_2\mathrm{cov}(T_2, X_1 \times T_1 + X_2 \times T_2)$$

$$\sigma_p^2 = X_1 \times \mathrm{cov}_{1p} + X_2 \times \mathrm{cov}_{2p}$$

$X_1 \times \mathrm{cov}_{1p}$ is the contribution of title 1 to the risk of the portfolio.

This contribution is based on the proportion invested in the title and risk of the portfolio measured by $\mathrm{cov}_{1p}/\sigma_p$.

This portfolio risk is measured based on Title 1 and portfolio covariation of the constitution of which it participates. Based on the formula that defines a security risk in a portfolio,

$$\frac{\mathrm{cov}_{1p}}{\sigma_p} = \frac{X_1 \times \sigma_1^2 + X_2 \times \mathrm{cov}_{12}}{\sigma_p}$$

we conclude:

- choosing a title for inclusion in a portfolio will not be based on its individual characteristics (σ_1), but on the behavior of the portfolio (cov_{1p});
- the risk of a title is not unique, it depends on the portfolio is included in.

▪ ***The determination of the absolute minimum variance of the portfolio structure with (PVMA) (case of the portfolio of two titles)***

The portfolio with minimum absolute variance or the optimal portfolio is that portfolio that ensures a maximum return while minimizing risk exposures.

Depending on the absolute minimum absolute variance portfolio it is possible to separate the multitude of portfolios into two distinct sections, as follows:

a) *Inefficient portfolios (mostly)* – are those portfolios below the PVMA level and any increase in risk is associated with a decrease in expected returns.
b) *Efficient portfolios (dominant)* – are those portfolios located above the PVMA level and that associates any risk increase with non-linear growth of expected returns.

C. Portfolio management that consists of two instruments on the capital market in our country

To study the applicability of the model, we have built a portfolio of securities issued by the two companies in our country and that are traded through Bucharest Stock Exchange. For this purpose, we used two of the 10 companies selected to be part of the portfolio (of 20 analyzed). The two companies chosen are OMV Petrom SA and Transylvania Bank.

Table 1. *Return securities included in the portfolio*

SNP	0.001669
TLV	0.001933

Estimated yield of the regarded portfolio is determined as follows:

$E_p = X_1 \times E_1 + X_2 \times E_2$

where:

X_1, X_2 = participation shares in portofolio;

E_1, E_2 = returns of the two titles;

$E_p = X_{SNP} \times E_{SNP} + X_{TLV} \times E_{TLV}$.

Simultaneously, we calculated the risk related to the investment, according to the equation:

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_2^2 \times \sigma_2^2 + 2 \times X_1 \times X_2 \times \sigma_{12}$$

$$\sigma_p^2 = X_{SNP}^2 \times \sigma_{SNP}^2 + X_{TLV}^2 \times \sigma_{TLV}^2 + 2 \times X_{SNP} \times X_{TLV} \times \sigma_{SNP/TLV}$$

The investor has the possibility to choose over the weights assigned to each type of action in virtual portfolio that will be created. The influence on the evolution of the share over the portfolios and risk is presented in the following table:

Table 2. *Company profitability and risk analysis for a portfolio of shares of SNP and TLV*

X_{SNP}	X_{TLV}	$X \times E_{SNP}$	$X \times E_{TLV}$	E_P	X^2_{SNP}	X^2_{TLV}	$X^2_{SNP} \times \sigma^2_{SNP}$	$X^2_{TLV} \times \sigma^2_{TLV}$
100.00%	0.00%	0.001669	0.000000	0.001669	1.00	0.00	0.000221	0.000000
90.00%	10.00%	0.0015021	0.0001933	0.001695	0.81	0.01	0.000179	0.000003
80.00%	20.00%	0.0013352	0.0003866	0.001722	0.64	0.04	0.000141	0.000013
70.00%	30.00%	0.0011683	0.0005799	0.001748	0.49	0.09	0.000108	0.000030
60.00%	40.00%	0.0010014	0.0007732	0.001775	0.36	0.16	0.000080	0.000053
50.00%	50.00%	0.0008345	0.0009665	0.001801	0.25	0.25	0.000055	0.000083
40.00%	60.00%	0.0006676	0.0011598	0.001827	0.16	0.36	0.000035	0.000120
30.00%	70.00%	0.0005007	0.0013531	0.001854	0.09	0.49	0.000020	0.000163
20.00%	80.00%	0.0003338	0.0015464	0.001880	0.04	0.64	0.000009	0.000212
10.00%	90.00%	0.0001669	0.0017397	0.001907	0.01	0.81	0.000002	0.000269

$2 \times X_{SNP} \times X_{TLV} \times \sigma_{SNP/TLV}$	σ^2_P	σ_P
0	0.000221	0.014866
0.00001998	0.000202	0.014224
0.00003552	0.000190	0.013793
0.00004662	0.000185	0.013594
0.00005328	0.000186	0.013637
0.0000555	0.000194	0.013919
0.00005328	0.000208	0.014428
0.00004662	0.000229	0.015139
0.00003552	0.000257	0.016026
0.00001998	0.000291	0.017062

The interpretation of previous results is concluded:

- in the case of the echiponderat portfolio, consisting of two titles, for example, was a daily return of 0.1801% with a standard deviation of 1.3919%, which means that future hoped profitability will most likely be equal with 0.1801% ± 1.3919% and will be in the range {-1.2118, 1.572};

- the maximum value of the return that can be obtained is 1.572%, which enjoys the investor preference for accepting securities with a risk as high volatility in the belief that it will earn above average profitability;
- Instead, an investor with adversity towards risks will not accept anything else than securities that records the highest return per unit of risk, or the reverse situation, when titles have the lowest risk per unit of forecast return.

D. Profitability and the risk of a portfolio composed of "n" financial titles

Based on the above methodology, it is possible to determine the yield and the risk for any type of financial instrument portfolio, regardless of the number of securities included in its structure.

If we extend, in a first phase, the previous analysis to a portfolio of three financial titles, we believe that this new title is a combination of portfolio included and the two financial instruments previously considered. This combination is particularly effective in the curve risk - gain and is closer to the vertical axis and the top of the chart (hazard minimum - maximum gain). As the correlation between titles is lower (negative), so the curve is inclined to the right.

The above elements can be represented graphically as follows:

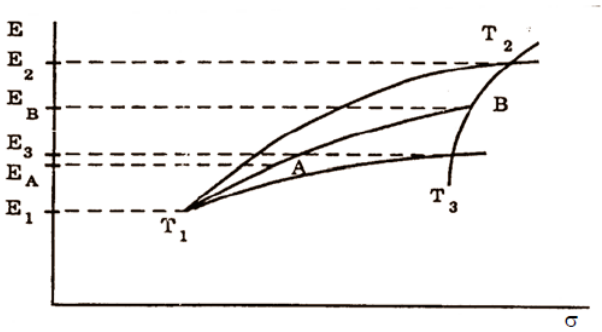

Figure 1. *The combination of three titles*

Based on the methodology presented elements, in the case of n titles portfolio, the profitability and risk can be determined as follows:

$$E_p = \sum_{i=1}^{n} X_i E_i$$

$$\sigma_p^2 = \sum_i X_i^2 \times \sigma_i + \sum_i \sum_j X_i \times X_j \times \mathrm{cov}_{ij}; \qquad i \neq j$$

or if we note cov_{ij} by σ_{ij} and $V_i = \sigma_i^2$ by σ_{ii} then:

$$\sigma_p^2 = \sum_i \sum_j X_i \times X_j \times \sigma_{ij}$$

As you can see, even in the portfolios consisting of "n" financial titles, the portfolio return depends on the expected return of each title included in the structure of the portfolio, and the share which they hold in this structure.

In the case of portfolio risk can be seen that its level is influenced both by:

- individual risks of each title included in the portfolio;
- share of the title in the portfolio;
- covariance of the returns of securities, taken two by two.

E. The determination of the optimum portfolio using the Markowitz model

The Markowitz model of portfolio diversification of financial instruments may lead to the identification of the optimal portfolio of risky titles, respectively of companies providing a maximum expected the return for a given level of risk that the capital investors are willing to assume due to their behavior towards risk.

From a mathematical perspective, the optimal portfolio will satisfy the following conditions:

$$\min \sigma_p = \left(\sum_i \sum_j X_i \times X_j \times \mathrm{cov}_{ij} \right)^2$$

under restrictions:

$$E_p = \sum_i X_i \times E_i = E_e$$

$$\sum_i X_i = 1.$$

In this case, using the Lagrange method, with which the optimal solution is determined as follows:

$$L = \sigma_p + \lambda_1 \left(E_e - \sum_i X_i \times E_i \right) + \lambda_2 \left(1 - \sum_i X_1 \right)$$

To obtain the optimal solution is resolved the system:

$$\frac{\delta L}{\delta \lambda_1} = E_e - \sum_i X_i \times E_i = 0$$

$$\frac{\delta L}{\delta \lambda_2} = 1 - \sum_i X_i = 0$$

$$\frac{\delta L}{\delta X_1} = \frac{\delta \sigma_p}{\delta X_1} - \lambda_1 \times E_1 - \lambda_2 = 0$$

....

$$\frac{\delta L}{\delta X_n} = \frac{\delta \sigma_p}{\delta X_n} - \lambda_1 \times E_n - \lambda_2 = 0$$

In which for whatever "i" and "j", it results:

$$\frac{\delta \sigma_p}{\delta X_n} - \lambda_1 \times E_i = \frac{\delta \sigma_p}{\delta X_n} - \lambda_1 \times E_j$$

The multiplier λ_1 measures the risk per unit change in yield variation, $\lambda_1 = \delta\sigma_e/\delta E_e$ that is the slope of the tangent to the efficient frontier at the point e.

Denote by:

$$s_e = \frac{\delta E_e}{\delta \sigma_e} \Longrightarrow \lambda_1 = \frac{1}{s_e} \Longrightarrow E_j - E_i = s_e \left(\frac{\delta \sigma_e}{\delta E_j} - \frac{\delta \sigma_e}{\delta E_i} \right)$$

Multiplying each term by X_{ie} and summing after i we will obtain:

$$\sum_i X_{ie}(E_j - E_i) = s_e \left(\sum_i X_{ie} \frac{\delta \sigma_e}{\delta E_j} - \sum_i X_{ie} \frac{\delta \sigma_e}{\delta E_i} \right) \Longrightarrow$$

$$E_j - \sum_i X_{ie} \times E_i = s_e \left(\frac{\delta \sigma_e}{\delta E_j} - \sum_i X_{ie} \frac{\delta \sigma_e}{\delta E_i} \right)$$

where:

$\sum_i X_{ie} \times E_i$ = contributions amount to yield securities in portfolio e (E_e)

$\frac{\delta \sigma_e}{\delta E_j}$ = the titles risk in portfolio e ($\frac{cov_{je}}{\sigma_e}$)

$\sum_i X_{ie} \frac{\delta \sigma_e}{\delta E_i}$ = the titles amount contributions at the risk of portfolio e (σ_e).

Finally, you get the relation according to which:

$E_j - E_e = s_e\left(\frac{cov_{ip}}{\sigma_e} - \sigma_e\right).$

This is the necessary but not sufficient condition of efficiency for a title to be held efficiently in a portfolio.

All combinations of all optimal portfolios with n titles in a market can be joined by a curve called the efficient frontier, a concept introduced by Harry Markowitz in his work entitled "Portfolio Selection" (1952). This curve has a convex shape by default. For any point inside the efficient frontier there is at least one portfolio on the market which can be put into correspondence a set of risk - gain variables. The portfolios from the efficient frontier have the following property: at a certain level of earning it has the minimum variance (low risk) or a certain level of risk it offers the maximum expected gain among all other variants of possible combinations.

The efficient frontier of a market is made up of the points of minimum variance of the curves related to risk - gain profiles for all combinations of titles on a market. This new concept associated with placements in financial markets is difficult to empirically highlight due to the risky number of titles that can be combined.

The action strategy of an investor is determined by the meeting between the desires set and the multitude of posibilities. The desires multitude consists of the indifference curves that are the expression of the investors preference in plan E - σ and results directly from its utility function. The multitude of possibilities is represented by the plane's efficiency frontier in plan E - σ which is obtained starting from the securities set of individual investor expectations. Only the efficient portfolios, however, are considered by the investor. The choice of the investor will be the corresponding point of tangency between the two curves. At this point the optimal portfolio is obtained.

The chosen portfolio will depend on the aversion degree of risk of the investor. If his aversion is strong, he will choose a portfolio located on the left of the efficient frontier corresponding to the weakest level of risk. With a weaker aversion he will choose a portfolio further to the right on the border.

According to the studies developed by Markowitz and continued later by Sharpe, the second stage involves the construction of an efficient portfolio of securities included in the portfolio without risk. Each portfolio has an associated straight line of capital allocation – DAC which starts at the point corresponding to the risk-free rate of the market. The investor will try a number of combinations until he will get the right combination for which the line allocation of capital is tangent to the efficient frontier. The point P in which the straight line DAC is tangent to the efficient frontier corresponds to the optimal portfolio that maximizes the earning rate of risk accepted ("reward - to - variability ratio"). This portfolio P is the final point in the construction of the optimal portfolio (effective) according to the theory of capital allocation (Markovitz).

F. The portfolio management consisting of three financial instruments on the capital market in our country

The portfolio composed of the two financial titles (shares issued by OMV Petrom SA and Transylvania Bank) previously analyzed may be varied by introducing in its structure a third financial instrument.

In this respect, the capital investor has the opportunity to purchase two categories of financial instruments as follows:

- securities issued by an economic entity;
- government securities.

- ***Efficiency and risk of a portfolio composed of three classes of shares***

In order to form this portfolio, we have included in its structure of financial instruments shares issued by the Financial Investment Company “SIF Muntenia S.A.”. Based on the above methodology we will calculate the yield and the risk attached to shares issued by the considered issuer.

$E_p = X_1 \times E_1 + X_2 \times E_2 + X_3 \times E_3$

$E_p = X_{SNP} \times E_{SNP} + X_{TLV} \times E_{TLV} + X_{SIF4} \times E_{SIF4}$

Further to this, we will determine the values for the correlation coefficients between the three financial instruments included in its structure:

$$\sigma_p^2 = \sum_i \sum_j X_i \times X_j \times \sigma_{ij}$$

$$\sigma_p^2 = X_1^2 \times \sigma_1^2 + X_2^2 \times \sigma_2^2 + X_3^2 \times \sigma_3^2 + 2 \times X_1 \times X_2 \times \sigma_{12} + 2 \times X_1 \times X_3 \times \sigma_{13} + 2 \times X_2 \times X_3 \times \sigma_{23}$$

$$\sigma_p^2 = X_{SNP}^2 \times \sigma_{SNP}^2 + X_{TLV}^2 \times \sigma_{TLV}^2 + X_{SIF4}^2 \times \sigma_{SIF4}^2 + 2 \times X_{SNP} \times X_{TLV} \times \sigma_{SNP/TLV} +$$

$$+ 2 \times \sigma_{SNP/SIF4} + 2 \times X_{TLV} \times X_{SIF4} \times \sigma_{TLV/SIF4}$$

Also in this case, the investor may choose the weights assigned to each type of shares in the portfolio built. The influence on the evolution of this share portfolio of return and risk is presented in the following table:

Table 3. *Analyse indicators of profitability and risk for the portfolio made of shares SNP, TLV and SIF4*

X_{SNP}	X_{TLV}	X_{SIF4}	$X_{SNP}\times E_{SNP}$	$X_{TLV}\times E_{TLV}$	$X_{SIF4}\times E_{SIF4}$	E_P	X^2_{SNP}
100.00%	0.00%	0.00%	0.001669	0.000000	0.000000	0.001669	1.000000
80.00%	10.00%	10.00%	0.001335	0.000193	0.000134	0.001662	0.640000
10.00%	80.00%	10.00%	0.000167	0.001546	0.000134	0.001847	0.010000
60.00%	20.00%	20.00%	0.001001	0.000387	0.000268	0.001656	0.360000
40.00%	30.00%	30.00%	0.000668	0.000580	0.000401	0.001649	0.160000
30.00%	40.00%	30.00%	0.000501	0.000773	0.000401	0.001675	0.090000
20.00%	40.00%	40.00%	0.000334	0.000773	0.000535	0.001642	0.040000
40.00%	20.00%	40.00%	0.000668	0.000387	0.000535	0.001589	0.160000
0.00%	50.00%	50.00%	0.000000	0.000967	0.000669	0.001636	0.000000
50.00%	0.00%	50.00%	0.000835	0.000000	0.000669	0.001504	0.250000
30.00%	10.00%	60.00%	0.000501	0.000193	0.000803	0.001497	0.090000
10.00%	30.00%	60.00%	0.000167	0.000580	0.000803	0.001550	0.010000
10.00%	20.00%	70.00%	0.000167	0.000387	0.000937	0.001490	0.010000
10.00%	10.00%	80.00%	0.000167	0.000193	0.001070	0.001431	0.010000
20.00%	0.00%	80.00%	0.000334	0.000000	0.001070	0.001404	0.040000
0.00%	20.00%	80.00%	0.000000	0.000387	0.001070	0.001457	0.000000
10.00%	0.00%	90.00%	0.000167	0.000000	0.001204	0.001371	0.010000
0.00%	10.00%	90.00%	0.000000	0.000193	0.001204	0.001398	0.000000
0.00%	0.00%	100.00%	0.000000	0.000000	0.001338	0.001338	0.000000
33.00%	33.00%	34.00%	0.000551	0.000638	0.000455	0.001644	0.108900

X^2_{TLV}	X^2_{SIF4}	$X^2_{SNP}\times\sigma^2_{SNP}$	$X^2_{TLV}\times\sigma^2_{TLV}$	$X^2_{SIF4}\times\sigma^2_{SIF4}$	$2\times X_{SNP}\times X_{TLV}\times \sigma_{SNP/TLV}$
0.000000	0.000000	0.000221	0.000000	0.000000	0.000000
0.010000	0.010000	0.000141	0.000003	0.000003	0.000018
0.640000	0.010000	0.000002	0.000212	0.000003	0.000018
0.040000	0.040000	0.000080	0.000013	0.000013	0.000027
0.090000	0.090000	0.000035	0.000030	0.000029	0.000027
0.160000	0.090000	0.000020	0.000053	0.000029	0.000027
0.160000	0.160000	0.000009	0.000053	0.000051	0.000018
0.040000	0.160000	0.000035	0.000013	0.000051	0.000018
0.250000	0.250000	0.000000	0.000083	0.000079	0.000000
0.000000	0.250000	0.000055	0.000000	0.000079	0.000000
0.010000	0.360000	0.000020	0.000003	0.000114	0.000007
0.090000	0.360000	0.000002	0.000030	0.000114	0.000007
0.040000	0.490000	0.000002	0.000013	0.000155	0.000004
0.010000	0.640000	0.000002	0.000003	0.000203	0.000002
0.000000	0.640000	0.000009	0.000000	0.000203	0.000000
0.040000	0.640000	0.000000	0.000013	0.000203	0.000000
0.000000	0.810000	0.000002	0.000000	0.000257	0.000000
0.010000	0.810000	0.000000	0.000003	0.000257	0.000000
0.000000	1.000000	0.000000	0.000000	0.000317	0.000000
0.108900	0.115600	0.000024	0.000036	0.000037	0.000024

$2\times X_{SNP}\times X_{SIF}\times \sigma_{SNP/SIF4}$	$2\times X_{TLV}\times X_{SIF4}\times \sigma_{TLV/SIF4}$	σ^2_P	σ_P
0.000000	0.000000	0.000221	0.014866
0.000016	0.000002	0.000183	0.013545
0.000002	0.000016	0.000253	0.015910
0.000024	0.000008	0.000164	0.012794
0.000024	0.000017	0.000162	0.012713
0.000018	0.000023	0.000169	0.013011
0.000016	0.000031	0.000177	0.013316
0.000032	0.000016	0.000164	0.012819
0.000000	0.000049	0.000211	0.014517
0.000050	0.000000	0.000184	0.013565
0.000036	0.000012	0.000191	0.013830
0.000012	0.000035	0.000200	0.014130
0.000014	0.000027	0.000216	0.014706
0.000016	0.000016	0.000242	0.015556
0.000032	0.000000	0.000243	0.015601
0.000000	0.000031	0.000247	0.015723
0.000018	0.000000	0.000277	0.016637
0.000000	0.000017	0.000278	0.016660
0.000000	0.000000	0.000317	0.017804
0.000022	0.000022	0.000165	0.012846

As it can be seen, the introduction to the portfolios structure of the third type of financial instruments, respectively the actions of SIF4 - Muntenia, led to an increase in the performance of the portfolio, simultaneously with a decrease in risk associated with this investment. This is due to the fact that the new shares in the portfolio are some with a specific lower-risk.

- **The yield and the risk of a portfolio composed of two categories of actions and a state title**

Another option of diversifying its portfolio of financial instruments is represented by the purchase of government bonds. They have the great advantage that, unlike the securities issued by business entities, they are not risk-bearing. To analyze this type of portfolio we have included state securities with a daily yield of 0.015836%. The interest rate for government bonds with interest was taken from the National Bank of Romania site www.bnro.ro, but because it is available as annual percentage rate, daily interest rate was obtained from the following formula:

$$(1 + \mathrm{rate}_{\mathrm{annual}}) = \left(1 + \mathrm{rate}_{\mathrm{daily}}\right)^{365}$$

Table 4. *Profitability of securities included in the portfolio*

SNP	0.001669
TLV	0.001933
TS	0.015836

Also, in this case, we simulated several options for allocating percentages of participation of the titles in the construction of the portfolio, the results being presented in the following table:

Table 5. *Analyse indicators of profitability and risk for the portfolio made of the shares of SNP, TLV and a state title*

X_{SNP}	X_{TLV}	X_{TS}	E_P	σ^2_P	σ_P
100.00%	0.00%	0.00%	0.001669	0.000221	0.014866
80.00%	10.00%	10.00%	0.003112	0.000183	0.013545
10.00%	80.00%	10.00%	0.003297	0.000253	0.015910
60.00%	20.00%	20.00%	0.004555	0.000164	0.012794
40.00%	30.00%	30.00%	0.005998	0.000162	0.012713
30.00%	40.00%	30.00%	0.006025	0.000169	0.013011
20.00%	40.00%	40.00%	0.007441	0.000177	0.013316
40.00%	20.00%	40.00%	0.007389	0.000164	0.012819
0.00%	50.00%	50.00%	0.008885	0.000211	0.014517
50.00%	0.00%	50.00%	0.008753	0.000184	0.013565
30.00%	10.00%	60.00%	0.010196	0.000191	0.013830
10.00%	30.00%	60.00%	0.010248	0.000200	0.014130
10.00%	20.00%	70.00%	0.011639	0.000216	0.014706
10.00%	10.00%	80.00%	0.013029	0.000242	0.015556
20.00%	0.00%	80.00%	0.013003	0.000243	0.015601
0.00%	20.00%	80.00%	0.013055	0.000247	0.015723
10.00%	0.00%	90.00%	0.014419	0.000277	0.016637
0.00%	10.00%	90.00%	0.014446	0.000278	0.016660
0.00%	0.00%	100.00%	0.015836	0.000317	0.017804
33.00%	33.00%	34.00%	0.006573	0.000165	0.012846

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