

## Land value dynamics with endogenous human and physical capital accumulation

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**Abstract.** *This study builds a three-sector growth model with endogenous physical and human wealth accumulation. It deals with dynamic interactions among land value, economic structural change, wealth accumulation and human capital growth. The economy consists of industrial, agricultural and education sectors. The model is based on the neoclassical growth theory, the Ricardian theory, and the Uzawa-Lucas model with Zhang's utility function. The human capital accumulation is through Arrow's learning by doing, Uzawa's learning through education, and Zhang's learning through consuming (leisure creativity). By simulation, we demonstrate that the economic system has a unique stable steady state. Comparative dynamic analysis is conducted with regard to changes in the propensity to receive education, the propensity to consume housing, the propensity to consume industrial goods, the propensity to consume agricultural goods, the propensity to save, and the output elasticity of land of the agricultural sector.*

**Keywords:** land value and rent, education and human capital, wealth accumulation, economic structure.

**JEL Classification:** E22, E24.

## 1. Introduction

Dynamics of land value is obviously an important issue in economics. It is also obvious that the dynamics is closely related to economic development in contemporary economies. Nevertheless, there are few theoretical (mathematical) models on dynamic interdependence between land value, economic growth, and economic structural change. The lacking of formal models on land value and economic growth is mainly due to the fact that it is not an easy matter to deal with the issue within a compact framework. The purpose of this study is to examine dynamic of land value in a three-sector growth model with economic structure. In this study the main mechanism of driving economic growth is through wealth and human capital accumulation. In *On the Principles of Political Economy and Taxation* published in 1817, Ricardo studied income distribution to explain how a change in this distribution could hinder or favor accumulation. The Ricardian system links wages, interest rate, and rent together in a compact theory. Ricardo distinguished between the three production factors, labor, capital, and land. He provided a theory to explain the functional income distribution of labor share, the capital, and the land rent share of total income. Ricardo (1821: preface) pointed out: "The produce ... is divided among three classes of the commodity, namely, the proprietor of land, the owners of the stock or capital necessary for its cultivation, and laborers by whose industry it is cultivated. But in different stages of the society, the proportions of the whole produce of the earth which will be allotted to each of these classes, under the names of rent, profits, and wages, will be essentially different; depending mainly on the actual fertility of the soil, on the accumulation of capital and population, and on the skill, ingenuity, and the instruments in agriculture." Since the publication of the *Principles*, economists have made a lot of efforts to extend or generalize the system (see Barkai, 1959, 1966; Pasinetti, 1960, 1974; Cochrane, 1970; Brems, 1970; Caravale and Tosato, 1980; Casarosa, 1985; Negish, 1989; Morishima, 1989). Nevertheless, what Ricardo described long time is still valid for the current state of the literature: "To determine the laws which regulate this distribution, is the principal problem in Political Economy: much as the science has been improved by the writings of Turgot, Stuart, Smith, Say, Sismondi, and others, they afford very little satisfactory information respecting the natural course of rent, profit, and wages." (Ricardo, 1821: preface). It should be remarked that in Ricardo's statement there is no reference to land value (price). In contemporary economies land is sold and bought by different agents. It is not proper to assume that land is owned by a special class. This study introduces land market through which land value is determined by competitive mechanism. In particular, we relate to land value to wage rate of interest, physical wealth value, and human capital value with endogenous wealth and human capital. The unique feature of this paper is to connect land value determination with investment in education and time distribution. Although the traditional Ricardian theory deals with value determination and income distribution, it fails to provide a proper microeconomic mechanism of wealth accumulation. The neoclassical growth theory based on the Solow one-sector growth model has been developed with endogenous wealth accumulation. This study integrate the basic economic mechanisms of the Ricardian and neoclassical growth theories within a compact analytical framework for studying dynamic interactions among growth, physical and human capital accumulation, income distribution, and economic structures.

Different from the time when Ricardo was constructing his theory of land distribution, in modern times housing markets are more important in determining land values. There are many empirical studies on land values for modern economies. Cho (1996: 145) observed, "During the past decade, the number of studies on intertemporal changes in house prices has increased rapidly because of wider availability of extensive micro-level data sets, improvements in modeling techniques, and expanded business applications." The literature on house and land prices has been increasingly expanding in recent years (e.g., Bryan and Colwell, 1982; Case and Quigley, 1991; Chinloy, 1992; Clapp and Giaccotto, 1994; Calhoun, 1995; Quigley, 1995; Capozza and Seguin, 1996; Alpanda, 2012; Du and Peiser, 2014; Kok *et al.* 2014). But there are only a few formal growth models with endogenous land values. As recently reviewed by Liu *et al.* (2011: 1), "Although it is widely accepted that house prices could have an important influence on macroeconomic fluctuations, quantitative studies in a general equilibrium framework have been scant." This study examines land prices in a general equilibrium framework with homogeneous population and heterogeneous goods.

Endogenous human capital in economic growth is currently a main topic in economic theory and empirical research. It has been argued by many economists that human capital is a key determinant of economic growth (Easterlin, 1981; Hanushek and Kimko, 2000; Barro, 2001; Krueger and Lindahl, 2001; Castelló-Climent and Hidalgo-Cabrillana, 2012; and Barro and Lee, 2013). Mincer (1974) published the seminal work in 1974 on estimating the impact of education on earnings. Earlier studies (e.g., Tilak, 1989) show that spread education can substantially reduce inequality within countries. Could *et al.* (2001) build a model to provide insights into the evolution of wage inequality within and between industries and education groups in the past few decades. Tselios (2008) studies the relationship between income and educational inequalities in the regions of the European Union, using the European Community Household Panel data survey for 94 regions over the period 1995-2000. The research finding suggests a positive relationship between income and educational inequalities. Fleisher *et al.* (2011) examine the role of education on worker productivity and firms' total factor productivity on the basis of firm-level data from China. A significant conclusion is that market mechanisms contribute to a more efficient use of human capital within firms. Zhu (2011) studies the individual heterogeneity in returns to education in China from 1995-2002. The study provides heterogeneous effects both within and between gender groups. There is also a large number of the theoretical literature on endogenous knowledge and economic growth (Romer, 1986; Lucas, 1988; Grossman and Helpman, 1991; and Aghion and Howitt, 1998). Especially the work by Lucas (1988) has caused a great interest in formal modeling of education and economic growth among economists. In fact, the first formal dynamic growth model with education was proposed by Uzawa (1965). Nevertheless, a main problem in the Uzawa-Lucas model and many of their extensions and generalizations is that all skills and human capital are formed due to formal schooling. It is argued that much of human capital may be accumulated in family and many other social and economic activities. Ignoring nonschool factors may make us misunderstand the role of formal education in economic development. In addition to formal schooling, this study takes account of Arrow's learning by doing (Arrow, 1962) and Zhang's creative leisure (Zhang, 2007) in modeling human capital accumulation. Another issue in modeling education and

economic growth is described by Chen and Chevalier (2008), “Making and exploiting an investment in human capital requires individuals to sacrifice not only consumption, but also leisure. When estimating the returns to education, existing studies typically weigh the monetary costs of schooling (tuition and forgone wages) against increased wages, neglecting the associated labor/leisure tradeoff.” This study makes time distribution between labor and education as endogenous variables. It should be noted that this study is a synthesis of Zhang’s recent two models (Zhang, 2013a, 2013b) with additional housing markets. This paper is organized as follows. Section 2 develops the growth model with endogenous physical and human capital accumulation with land distribution and housing. Section 3 examines dynamic properties of the model and simulates the model. Section 4 carries out comparative dynamic analysis with regard to changes in the propensity to receive education, the propensity to consume housing, the propensity to consume industrial goods, the propensity to consume agricultural goods, the propensity to save, and the output elasticity of land of the agricultural sector. Section 5 concludes the study. The appendix proves the results in section 3.

## 2. The model

We are concerned with dynamics of an economy with three, industrial, agricultural and education sectors. The industrial sector produces goods, which are freely traded in market. The agricultural sector produces agricultural goods, which is used for consumption. The industrial production is the same as that in Solow’s one-sector neoclassical growth model. It is a commodity used both for investment and consumption. Capital accumulation is endogenous. The education sector supplies education and output is human capital accumulation. Most aspects of the production sectors are similar to the neoclassical growth models (Uzawa, 1965; Burmeister and Dobell 1970; Azariadis, 1993; and Barro and Sala-i-Martin, 1995). The education sector is based on the Uzawa-Lucas two-sector models (Uzawa, 1965; Lucas, 1988). We select industrial goods to serve as numeraire. The population  $\bar{N}$  is homogenous and constant. The total qualified labor force is changeable as the household chooses time distribution between work and study and the household’s human capital is changeable. Households own assets of the economy and distribute their incomes to receive education, to consume and to save. Firms use labor and physical capital inputs to supply goods and services. Exchanges take place in perfectly competitive markets. Factor markets work well and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production, labor, managerial skill and capital ownership. Let prices be measured in terms of the commodity and the price of the commodity be unit. We use  $H(t)$  to stand for the level of human capital. The total land  $L$  is homogenous and constant. The land is owned by households and is distributed between housing and agricultural production in free land market. Households achieve the same utility level regardless of what profession they choose.

We use subscript index,  $i$ ,  $a$ , and  $e$ , to stand for industry, agriculture, and education, respectively. We use  $N_m(t)$  and  $K_m(t)$  to stand for the labor force and capital stocks employed by sector  $m$ ,  $i$ ,  $a$ , and  $e$ . Let  $T(t)$  and  $T_e(t)$  stand for, respectively, the work time and study time of a typical worker. The variable  $N(t)$  represents the total qualified labor force. A worker's labor force is  $T(t)H^m(t)$ , where  $m$  is a parameter measuring utilization efficiency of human capital. The labor input is the work time by the effective human capital. The total labor input is

$$N(t) = T(t)H^m(t)\bar{N}, \quad (1)$$

### The industrial sector

We now describe behavior of the three sectors. For the two production sectors, we use the neoclassical production functions. Let  $F_m(t)$  stand for the production function of sector  $m$ ,  $q = i, a, e$ . We assume that production is to combine labor force,  $N_i(t)$ , and physical capital,  $K_i(t)$ . We use the conventional production function to describe a relationship between inputs and output. The production function  $F_i(t)$  is specified as follows

$$F_i(t) = A_i K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \quad \alpha_i, \beta_i > 0, \quad \alpha_i + \beta_i = 1, \quad (2)$$

where  $A_i$ ,  $\alpha_i$ , and  $\beta_i$  are positive parameters. Markets are competitive; thus labor and capital earn their marginal products. The rate of interest  $r(t)$  and wage rate  $w(t)$  are determined by markets. The marginal conditions for the capital goods sector are

$$r(t) + \delta_k = \frac{\alpha_i F_i(t)}{K_i(t)}, \quad w(t) = \frac{\beta_i F_i(t)}{N_i(t)}, \quad (3)$$

where  $\delta_k$  is the fixed depreciation rate of physical capital.

### The agricultural sector

We assume that agricultural production is carried out by combination of capital  $K_a(t)$ , labor force  $N_a(t)$ , and land  $L_a(t)$  as follows

$$F_a(t) = A_a K_a^{\alpha_a}(t) N_a^{\beta_a}(t) L_a^{\zeta}(t), \quad A_a, \alpha_a, \beta_a, \zeta > 0, \quad \alpha_a + \beta_a + \zeta = 1, \quad (4)$$

where  $L_a(t)$  is the land employed by the agricultural sector, and  $A_a$ ,  $\alpha_a$ ,  $\beta_a$ , and  $\zeta$  are parameters. The marginal conditions are given by

$$r(t) + \delta_k = \frac{\alpha_a p_a(t) F_a(t)}{K_a(t)}, \quad w(t) = \frac{\beta_a p_a(t) F_a(t)}{N_a(t)}, \quad R(t) = \frac{\zeta p_a(t) F_a(t)}{L_a(t)}, \quad (5)$$

where  $p_a(t)$  is the price of agricultural goods and  $R(t)$  is the land rent.

### The education sector

The education sector is the same as in Zhang (2013b). It is characterized of perfect competition. Students pay the education fee  $p_e(t)$  per unit of time. The education sector pays teachers and capital with the market rates. The total education service is measured by the total education time received by the population. We specify the production function of the education sector as follows

$$F_e(t) = A_e K_e^{\alpha_e}(t) N_e^{\beta_e}(t), \quad \alpha_e, \beta_e > 0, \quad \alpha_e + \beta_e = 1, \quad (6)$$

where  $A_e$ ,  $\alpha_e$  and  $\beta_e$  are positive parameters. Let  $p_e(t)$  stand for the price of per unit time of education. The marginal conditions for the education sector are

$$r(t) + \delta_k = \frac{\alpha_e p_e(t) F_e(t)}{K_e(t)}, \quad w(t) = \frac{\beta_e p_e(t) F_e(t)}{N_e(t)}. \quad (7)$$

The demand for labor force from the education sector increases in the price and level of human capital and decreases in the wage rate.

### Choice between physical wealth and land

Structure of land ownership varies between economies. It may be owned by a small group of the population or publicly owned. In many modern economies the ownership structure is mixed in the sense that the state owns some part of the land and private individuals own some (subject to different forms of taxation). There are different approaches with regard to determination of land prices and rents. In some approaches (Iacoviello, 2005; Iacoviello and Neri, 2010) households are assumed to be credit constrained and these households use land or houses as collateral to finance consumption expenditures. These models with credit-constrained households are used to explain positive co-movements between house prices and consumption expenditures (see also, Campbell and Mankiw, 1989; Zeldes, 1989; Case, *et al.*, 2005; Mian and Sufi, 2010; Oikarinen, 2014; and Liu *et al.*, 2011). In this study, we assume that land is owned only by households. Land can be sold and bought in free markets without any friction and transaction costs. Land use will not waste land and land cannot regenerate itself. Households own land and physical wealth. We use  $p_L(t)$  to denote the price of land. As we assume capital and land markets to be at competitive equilibrium at any point in time, two options must yield equal returns, i.e.

$$\frac{R(t)}{p_L(t)} = r(t). \quad (8)$$

We may also explain why this equation holds. Consider now an investor with one unity of money. He can either invest in capital good thereby earning a profit equal to the net own-rate of return  $r(t)$  or invest in land thereby earning a profit equal to the net own-rate of return  $R(t)/p_L(t)$ . This equation enables us to determine choice between owning land and wealth. Indeed, the assumption is made under many strict conditions. For instance, we neglect any transaction costs and any time needed for buying and selling. Expectations on land by different individuals are extremely complicated. Equation (8) also implies perfect information and rational expectation.

### Consumer behaviors and wealth dynamics

Consumers make decisions on choice of consumption levels of education, services and commodities as well as on how much to save. In our approach consumers decide consumption levels of industrial and agricultural goods and lot size. They also decide how long to study, and how much to save. We use the approach to consumers' behavior proposed by Zhang (1993). We denote respectively physical wealth by  $\bar{k}(t)$  and land  $\bar{l}(t)$  owned by the representative household. The total value of wealth owned by the household  $a(t)$  is the sum of the two assets

$$a(t) = \bar{k}(t) + p_L(t)\bar{l}(t). \quad (9)$$

The household's current income  $y(t)$  consists of the interest payment  $r(t)\bar{k}(t)$ , the wage payment  $H^m(t)T(t)w(t)$ , and the land rent  $R(t)\bar{l}(t)$ . is given by

$$y(t) = r(t)\bar{k}(t) + H^m(t)T(t)w(t) + R(t)\bar{l}(t). \quad (10)$$

We call  $y(t)$  the current income. The per capita disposable income is given by

$$\hat{y}(t) = y(t) + a(t). \quad (11)$$

From (10) and (11) we also have

$$\hat{y}(t) = (1 + r(t))a(t) + H^m(t)T(t)w(t). \quad (12)$$

Education has been modeled in different ways (Becker, 1981; Cox, 1987; Behrman *et al.* 1982; Fernandez and Rogerson, 1998; Banerjee, 2004; Florida, et al. 2008; Galindez, 2011). The disposable income is used for saving, consumption, and education. In this study, we follow Zhang (2013b) in modeling choice of education time. The available time is distributed between education and work. At each point in time, a consumer would distribute the total available budget between saving  $s(t)$ , consumption of industrial goods  $c_i(t)$ , consumption of agricultural goods  $c_a(t)$ , education  $T_e(t)$ , and lot size  $l_h(t)$ . The budget constraint is given by

$$c_i(t) + s(t) + p_e(t)T_e(t) + p_a(t)c_a(t) + R(t)l_h(t) = - (1 + r(t))a(t) + H^m(t)T(t)w(t). \quad (13)$$

The time constraint for everyone is

$$T(t) + T_e(t) = T_0, \quad (14)$$

where  $T_0$  is the total available time. Substituting (14) into (13) yields

$$c_i(t) + s(t) + \bar{p}_e(t)T_e(t) + p_a(t)c_a(t) + R(t)l_h(t) = \bar{y}(t), \quad (15)$$

where

$$\bar{p}_e(t) \equiv p_e(t) + H^m(t)w(t), \quad \bar{y}(t) \equiv (1 + r(t))a(t) + H^m(t)T_0 w(t).$$

Following Zhang (2013), we introduce education into utility function. As education increases human capital, a rise in education tends to result in higher wages (e.g., Heckman, 1976; Lazear, 1977; Malchow-Møller, *et al.* 2011). As Lazear (1977: 570) points out: “education is simply a normal consumption good and that, like all other normal goods, an increase in wealth will produce an increase in the amount of schooling purchased. Increased incomes are associated with higher schooling attainment as the simple result of an income effect.” Education also brings about direct pleasure, more knowledgeable, higher social status and so on. The representative household has five variables,  $s(t)$ ,  $c_i(t)$ ,  $c_a(t)$ ,  $T_e(t)$ , and  $l_h(t)$ , to decide. The consumer’s utility function is specified as follows

$$U(t) = T_e^{\kappa_0}(t)c_i^{\xi_0}(t)c_a^{\mu_0}(t)l_h^{\eta_0}(t)s^{\lambda_0}(t), \quad \kappa_0, \mu_0, \eta_0, \lambda_0 > 0,$$

in which  $\kappa_0$ ,  $\xi_0$ ,  $\mu_0$ ,  $\eta_0$ , and  $\lambda_0$  are the household’s elasticity of utility with regard to education, industrial goods, agricultural goods, housing, and saving. We call  $\kappa_0$ ,  $\xi_0$ ,  $\mu_0$ ,  $\eta_0$ , and  $\lambda_0$  propensities to receive education, to consume industrial goods, to consume agricultural goods, to have housing, and to hold wealth, respectively. Maximizing  $U(t)$  subject to (15) yields

$$\begin{aligned} \bar{p}_e(t)T_e(t) &= \kappa \bar{y}(t), \quad c_i(t) = \xi \bar{y}(t), \quad p_a(t)c_a(t) = \mu \bar{y}(t), \quad R(t)l_h(t) = \eta \bar{y}(t), \\ s(t) &= \lambda \bar{y}(t), \end{aligned} \quad (16)$$

where

$$\kappa \equiv \rho \kappa_0, \quad \xi \equiv \rho \xi_0, \quad \mu \equiv \rho \mu_0, \quad \eta \equiv \rho \eta_0, \quad \lambda \equiv \rho \lambda_0, \quad \rho \equiv \frac{1}{\kappa_0 + \xi_0 + \mu_0 + \eta_0 + \lambda_0}.$$

### Wealth accumulation

According to the definition of  $s(t)$ , the change in the household’s wealth is given by



$$\dot{a}(t) = s(t) - a(t). \quad (17)$$

The equation simply states that the change in wealth is equal to saving minus dissaving.

### Dynamics of human capital

We assume that there are three sources of improving human capital, through education, “learning by producing”, and “learning by leisure”. Arrow (1962) first introduced learning by doing into growth theory; Uzawa (1965) took account of trade-offs between investment in education and capital accumulation, and Zhang (2007) introduced impact of consumption on human capital accumulation (via the so-called creative leisure) into growth theory. Following Zhang (2007), we propose that human capital dynamics is given by

$$\dot{H}(t) = \frac{\nu_e F_e^{a_e}(t) (H^m(t) T_e(t) \bar{N})^{b_e}}{H^{\pi_e}(t) \bar{N}} + \frac{\nu_i F_i^{a_i}(t)}{H^{\pi_i}(t) \bar{N}} + \frac{\nu_h C^{a_h}(t)}{H^{\pi_h}(t) \bar{N}} - \delta_h H(t), \quad (18)$$

where  $\delta_h (> 0)$  is the depreciation rate of human capital,  $\nu_e, \nu_i, \nu_h, a_e, b_e, a_i,$  and  $a_h$  are non-negative parameters. The signs of the parameters  $\pi_{je}, \pi_{ji},$  and  $\pi_{jh}$  are not specified as they may be either negative or positive. The above equation is a synthesis and generalization of Arrow’s, Uzawa’s, and Zhang’s ideas about human capital accumulation. The term

$$\frac{\nu_e F_e^{a_e}(t) (H^m(t) T_e(t) \bar{N})^{b_e}}{H^{\pi_e}(t) \bar{N}}$$

describes the contribution to human capital improvement through education (e.g., Uzawa, 1965; Lucas, 1988; Barro and Sala-i-Martin, 1995; and Solow, 2000). Human capital tends to increase with an increase in the level of education service,  $F_e$ , and in the (qualified) total study time,  $H^m T_e \bar{N}$ . The population  $\bar{N}$  in the denominator measures the contribution in terms of per capita. The term  $H^{\pi_e}$  indicates that as the level of human capital of the population increases, it may be more difficult (in the case of  $\pi_e$  being large) or easier (in the case of  $\pi_e$  being small) to accumulate more human capital via formal education. We take account of learning by producing effects in human capital accumulation by the term  $\nu_i F_i^{a_i} / H^{\pi_i} \bar{N}$ . We take account of learning by consuming by the term  $\nu_h C^{a_h} / H^{\pi_h} \bar{N}$ . This term can be interpreted similarly as the term for learning by producing.

### Demand of and supply for education

The demand for education is  $T_e(t) \bar{N}$  and the supply of education service is  $F_e(t)$ . The condition that the demand for and supply of education balances at any point of time implies

$$T_e(t)\bar{N} = F_e(t). \quad (19)$$

### **Balances of demand and supply for agricultural goods**

The demand and supply for the agricultural sector's output balance at any point in time

$$C_a(t) = c_a(t)\bar{N} = F_a(t). \quad (20)$$

### **All the land owned by households**

The land owned by the population is equal to the national available land

$$\bar{l}(t)\bar{N} = L. \quad (21)$$

The assumption of fixed land is also a strict requirement. As observed by Glaeser, *et al.* (2005), land supply elasticity varies substantially over space in the USA (see also, Davis and Heathcote, 2007). This study neglects possible changes in land supply.

### **Full employment of capital**

We use  $K(t)$  to stand for the total capital stock. We assume that the capital stock is fully employed. We have

$$K_i(t) + K_a(t) + K_e(t) = K(t). \quad (22)$$

### **The value of physical wealth and capital**

The value of physical capital is equal to the value of physical wealth

$$\bar{k}(t)\bar{N} = K(t). \quad (23)$$

### **Full employment of labor force**

We assume that labor force is fully employed

$$N_i(t) + N_a(t) + N_e(t) = N(t). \quad (24)$$

### **The land market clearing condition implies**

The land is fully used

$$l_h(t)\bar{N} + L_a(t) = L. \quad (25)$$

We completed the model. The model is structurally general in the sense that some well-known models in economics can be considered as its special cases. For instance, if we fix wealth and human capital, then the model is a Walrasian general equilibrium model with a single type of households. Our model is structurally similar to the neoclassical growth model by Solow (1956) and Uzawa (1961). The model is built on the basis of Arrow's learning by doing and the two-sector growth model with endogenous education by Uzawa and Lucas. We now examine dynamics of the model.

### 3. The dynamics and the motion by simulation

The economic system contains many variables and these variables are nonlinearly related. As it is almost impossible to generally solve the model, we simulate the model. In the appendix, we show that the dynamics of the national economy can be expressed as two differential equations. First, we introduce a variable  $z(t)$  by

$$z(t) \equiv \frac{r(t) + \delta_k}{w(t)}.$$

The following lemma shows that the dynamics can be expressed by two dimensional differential equations. The lemma also provides a computational procedure to simulate the model.

#### Lemma

The motion of the system is determined by the following two differential equations with  $z(t)$  and  $H(t)$  as the variables

$$\begin{aligned} \dot{z}(t) &= \Lambda(z(t), H(t)), \\ \dot{H}(t) &= \Omega(z(t), H(t)), \end{aligned} \tag{26}$$

where are functions of  $z(t)$  and  $H(t)$  given in the appendix. The other variables of the economic system are determined uniquely by the following procedure:  $r(t)$  and  $w(t)$  by (A2)  $\rightarrow \bar{k}(t)$  by (A25)  $\rightarrow K_a(t)$  by (A19)  $\rightarrow K_i(t)$  and  $K_e(t)$  by (A22)  $\rightarrow N_i(t)$ ,  $N_e(t)$ , and  $N_a(t)$  by (A1)  $\rightarrow N(t)$  by (A16)  $\rightarrow p_e(t)$  by (A13)  $\rightarrow \bar{y}(t)$  by (A13)  $\rightarrow R(t)$  by (A11)  $\rightarrow p_L(t)$  by (A13)  $\rightarrow a(t)$  by (A26)  $\rightarrow L_a$  and  $l_h$  by (A9)  $\rightarrow p_a(t)$  by (A5)  $\rightarrow T_e(t)$ ,  $c_i(t)$ ,  $c_a(t)$ , and  $s(t)$  by (16)  $\rightarrow T(t)$  by (14)  $\rightarrow C_a(t)$  by (20)  $\rightarrow C_i(t) = c_i(t)\bar{N} \rightarrow p_L(t)$  by (A23)  $\rightarrow p_a(t)$  by (A5)  $\rightarrow F_i(t)$  by (2)  $\rightarrow F_a(t)$  by (4)  $\rightarrow F_e(t)$  by (6).

The lemma shows that once we determine the value of the variable with an initial condition, we can determine all the variables in the economic system. The lemma is important as it gives a procedure to follow the motion of the system with computer. As the expressions of the analytical results are tedious, we now specify the parameter values and simulate the model. We specify the parameters as follows

$$\begin{aligned} \bar{N} &= 10, \quad T_0 = 10, \quad L = 100, \quad m = 0.6, \quad \alpha_i = 0.29, \quad \alpha_e = 0.3, \quad \alpha_a = 0.1, \\ \beta_a &= 0.2, \quad A_i = 1, \\ A_a &= 0.5, \quad A_e = 1, \quad \lambda_0 = 0.8, \quad \xi_0 = 0.07, \quad \mu_0 = 0.04, \quad \eta_0 = 0.07, \\ \kappa_0 &= 0.01, \quad v_e = 0.8, \end{aligned}$$

$$\begin{aligned}
v_i &= 0.1, \quad v_h = 0.2, \quad a_e = 0.3, \quad b_e = 0.5, \quad a_i = 0.4, \quad a_h = 0.1, \quad b_e = 0.5, \\
\pi_e &= 0.5, \\
\pi_i &= 0.5, \quad \pi_h = 0.3, \quad \delta_k = 0.05, \quad \delta_h = 0.03.
\end{aligned} \tag{27}$$

The population is fixed at 10 and the land is 1. We assume that the propensity to save is much higher than the propensity to consume industrial goods and the propensity to consume agricultural goods. As shown in the appendix, the following variables are invariant in time

$$l_h = 7.14, \quad L_a = 28.5, \quad \bar{l} = 1.$$

We specify the following initial conditions

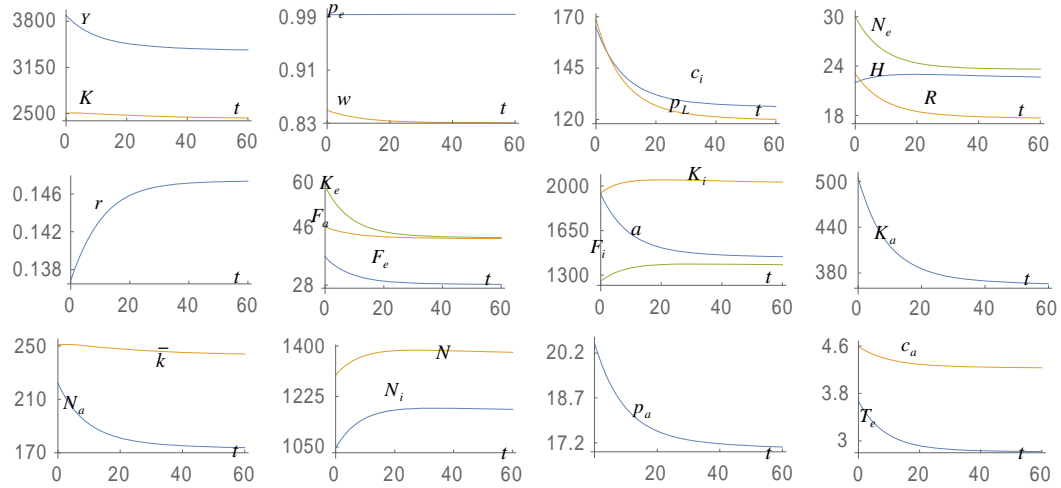
$$z(0) = 0.22, \quad H(0) = 22.$$

We plot the motion of the variables in Figure 1. In Figure 1 the national gross product (GDP) is

$$Y(t) = F_i(t) + p_a(t)F_a(t) + p_e(t)F_e(t) + l_h \bar{N} R(t).$$

The GDP and national capital stock fall over time till they become stationary. The wage rate, the price of land, price of agricultural goods, and land rent fall over time. The price of education is slightly affected. The rate of interest is increased. The human capital is increased. Although the education time is reduced, the total labor supply is increased.

The output levels and labor and capital inputs of both the education and the agricultural sectors fall over time. The output level and labor and capital inputs of the industrial sector rise over time. The household works more hours and studies less. The physical wealth, total wealth and consumption levels fall. It should be noted that the dynamic relationship between the GDP and the land price plotted in Figure 1 predicts the same as what is observed Liu *et al.* (2011: 1): “The recent financial crisis caused by a collapse of the housing market propelled the U.S. economy into the Great Recession. A notable development during the crisis period was a slump in business investment in tandem with a sharp decline in land prices.” The conclusions made by Liu *et al.* are based on the data for the Great Recession period as well as for the entire sample period from 1975 to 2010. Our comparative dynamic analysis in the rest of the paper also shows similar conclusions.

**Figure 1.** *The Motion of the Economic System*

From Figure 1 we observe that all the variables tend to become stationary in the long term. This implies the existence of some equilibrium point. We confirm the existence of equilibrium point as follows

$$Y = 3370.7, \quad K = 2423.1, \quad H = 22.45, \quad w = 0.83, \quad p_L = 118.95,$$

$$R = 17.54, \quad r = 0.15,$$

$$p_a = 16.96, \quad p_e = 0.99, \quad F_a = 42.22, \quad F_i = 1374, \quad K_e = 28.11,$$

$$K_a = 362.57, \quad K_i = 2018,$$

$$N_e = 23.56, \quad N_a = 172.4, \quad N_i = 1174.4, \quad \bar{k} = 242.3, \quad c_a = 4.22,$$

$$c_i = 125.3, \quad a = 1431.8,$$

$$T_e = 2.81.$$

(20)

The eigenvalues at the equilibrium point are

$$-0.114, \quad -0.029.$$

This guarantees the stability of the steady state. The stability is important as it implies that we can conduct comparative dynamic analysis.

#### 4. Comparative dynamic analysis

We now examine effects of changes in some parameters on the motion of the economic system. As the lemma gives a computational procedure to calibrate the motion of all the variables and the equilibrium point is locally stable, it is straightforward to conduct

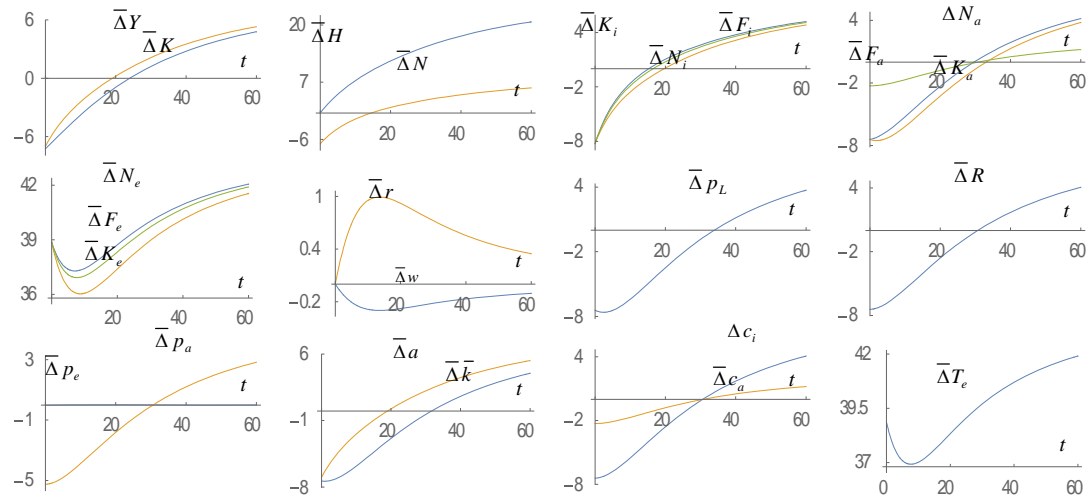
comparative dynamic analysis. In the rest of this study we use  $\bar{\Delta}x_j(t)$  to stand for the change rate of the variable,  $x_j(t)$ , in percentage due to changes in a parameter value.

### The propensity to receive education being enhanced

First we allow the propensity to receive education to be enhanced as follows:  $\kappa_0 : 0.01 \Rightarrow 0.015$ . The lot size and agricultural land-use are not affected,  $\bar{\Delta}l_h = \bar{\Delta}L_a = 0$ .

The time to receive education is augmented as the household changes the preference. The price of education is almost not affected. The human capital is increased. The total labor supply falls initially and rises in the long term. In the short run, the time shifted to education reduces the labor supply, while the rise in human capital is large. In the long term the rise due to improved human capital is larger than the falling in labor supply to reduced work hours. The education sector raises its inputs and output level. The GDP and national physical wealth fall initially and rise in the long term. The price of agricultural goods falls initially and rises in the long term. The output and capital and labor inputs of the two sectors are increased. The land value and rent are reduced initially and are enhanced in the long term. The wage rate  $w$  is lowered. It should be noted that the wage income  $H^m T w$  is increased. The rate of interest is increased. The household's physical wealth and total wealth and consumption levels of two goods are reduced initially and are augmented in the long term. In sum, as the household increases the propensity to receive education, the household's economic conditions worsen initially and become improved in the long term.

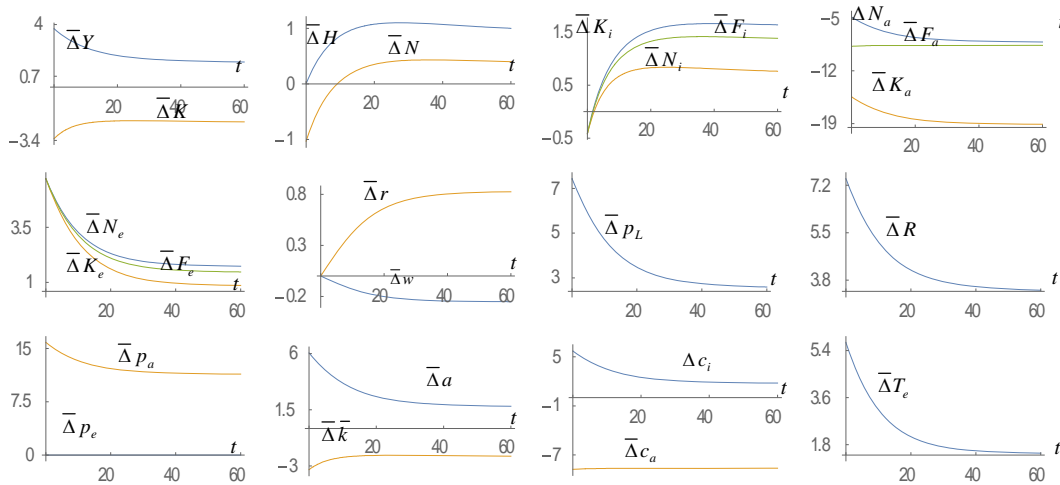
**Figure 2.** The Propensity to Receive Education Being Enhanced



### A rise in the total productivity factor of the education sector

We now examine effects of the following rise in the total factor productivity of the education sector:  $A_e: 1 \Rightarrow 1.1$ . The productivity change has no impact on the land distribution. The price of education is slightly increased in association of rising study hours. The human capital is increased. The total labor supply falls initially and then is increased. The total capital is lowered. The GDP is augmented. The wage rate is reduced and the rate of interest is increased. The land rent, land value and price of agricultural goods are enhanced. The output and capital and labor inputs of the agricultural sector are reduced. The output and capital and labor inputs of the industrial sector are reduced initially and raised in the long term. The household's total wealth is enhanced and the physical wealth is reduced. The consumption level of agricultural goods is reduced and the consumption level of industrial goods is increased.

**Figure 3.** A Rise in the Total Productivity Factor of the Education Sector



### The propensity to consume housing being enhanced

We allow the propensity to consume housing to be enhanced as follows:  $\eta_0: 0.07 \Rightarrow 0.08$ .

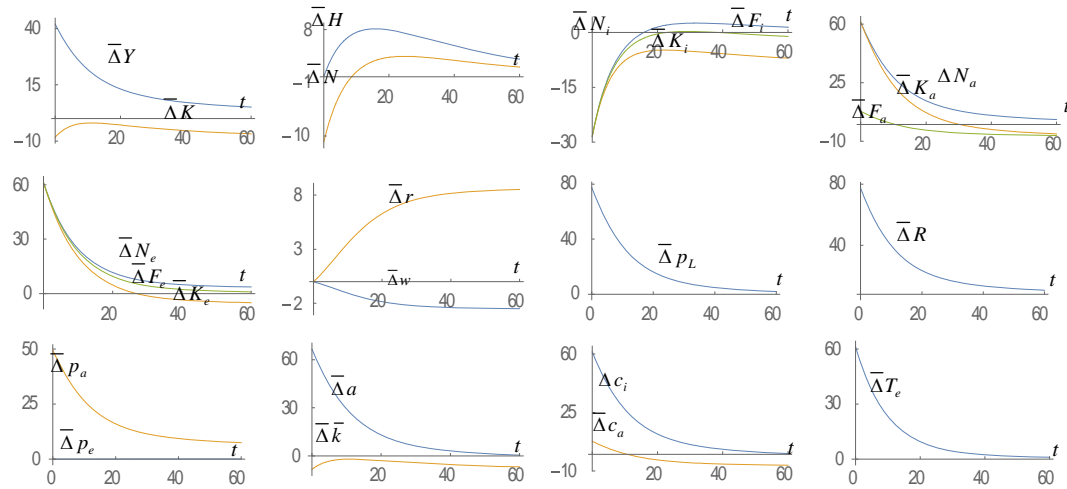
The lot size and agricultural land-use are changed as follows

$$\bar{\Delta}l_h = 3.7, \quad \bar{\Delta}L_a = -9.26.$$

More land is used for housing. The price of agricultural goods is increased. Initially the price is increased much. In association with the price change the agricultural sector initially attracts more capital and labor inputs. In initial stage even as the economy use more land for residential use, the agricultural sector produces more. But after a short period capital input is reduced. The net-long-run output level of the agricultural sector is reduced. The land value is augmented greatly in the initial stage and is slightly increased in the long term. The preference change reduces the capital and labor inputs, and the output level of the industrial sector. In the long term the output is slightly affected, the labor input is slightly increased, and the capital input is reduced. The time to receive education is raised initially and is

affected slightly in the long term. The price of education is almost not affected. The education sector raises its inputs and output levels in the initial stage. In the long term the output is slightly affected, the labor input is slightly increased, and the capital input is reduced. The wage rate is lowered. The household's physical wealth per capita is reduced. The household's total wealth is enhanced initially and changed slightly in the long term. In the long term the household's consumption level of industrial goods is slightly affected and consumption level of agricultural goods is reduced.

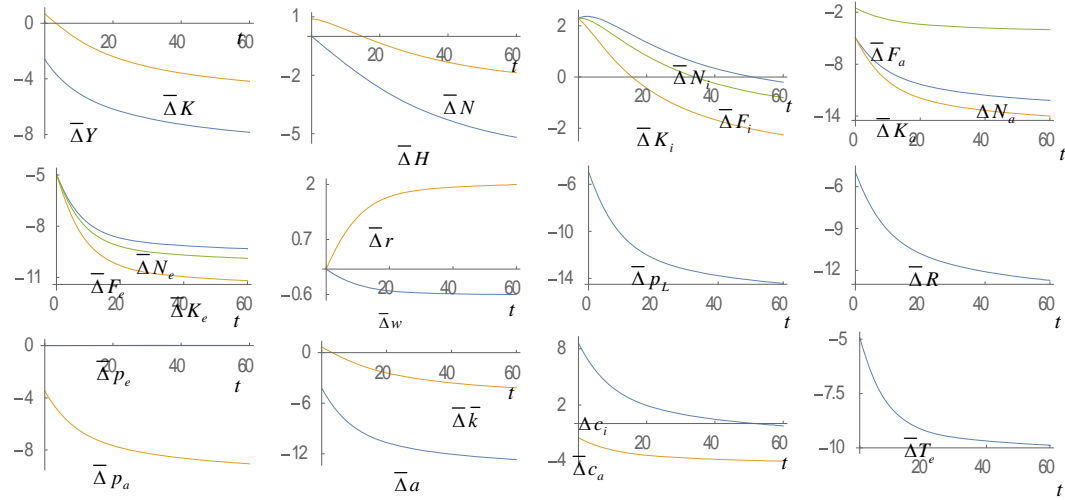
**Figure 4.** *The Propensity to Consume Housing Being Enhanced*



### The propensity to consume industrial goods being enhanced

We now examine the effects that the propensity to consume industrial goods is raised in the following way:  $\xi_0 : 0.07 \Rightarrow 0.08$ . The land distribution is not affected. As the household spends more out of the disposable income on consuming industrial goods, the total capital stock and the GDP are reduced. The wage rate is reduced and the rate of interest is enhanced. The household spends less time on education. The human capital falls. The price of education is slightly affected. The education output and two input factors of the education sector are reduced. The total labor supply is raised initially and reduced in the long term. Both the land value and the land rent are reduced. The household holds less total wealth and physical wealth. The household's consumption level of industrial goods rises and is slightly affected in the long term. The household's consumption level of agricultural goods falls. The price of agricultural goods falls in association with falling output of the sector. The output of the industrial sector is increased initially and reduced in the long term. The capital inputs of the two sectors are reduced in the long term.



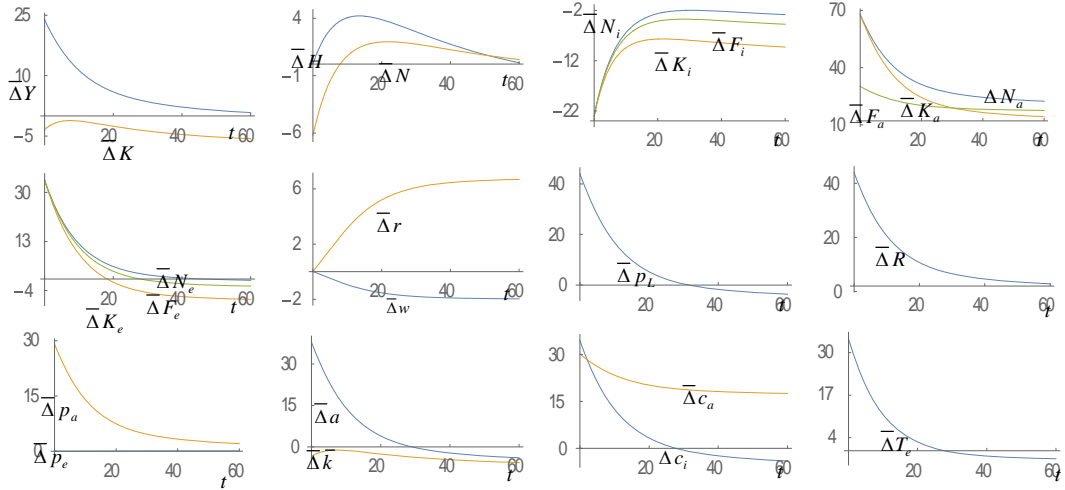
**Figure 5.** *The Propensity to Consume Industrial Goods Being Enhanced*

### A rise in the propensity to consume agricultural goods

We now study the effects that the propensity to consume agricultural goods is increased as follows:  $\mu_0 : 0.04 \Rightarrow 0.05$ . The land is redistributed between the residential use and agricultural production as follows

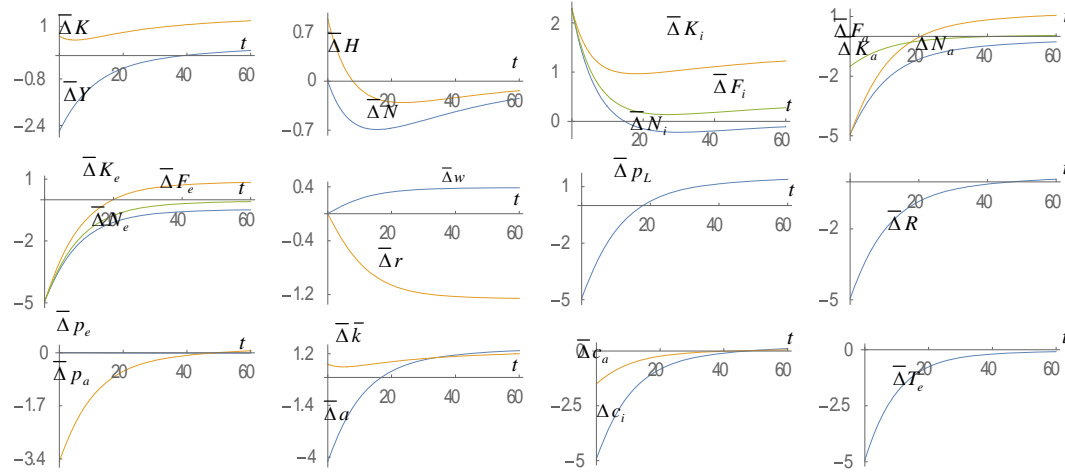
$$\bar{\Delta} l_h = -6.7, \quad \bar{\Delta} L_a = -16.67.$$

As the household spends more out of the disposable income on consuming agricultural goods, the total capital stock is reduced. The GDP is augmented. The household spends more hours on education. The human capital is increased. The total labor supply is reduced initially and is increased in the long term. The output and capital and labor inputs of the agricultural sector are enhanced. The output and capital and labor inputs of the agricultural sector are reduced. The output and capital and labor inputs of the education sector augmented initially and reduced in the long term. The wage rate is reduced and the rate of interest is enhanced. The land rent is increased. The value of land is increased initially and is reduced in the long term. The household holds more physical wealth. The household has more wealth initially and less wealth in the long term. The price of agricultural goods rises. The household's consumption level of agricultural goods rises. The household's consumption level of industrial goods rises initially and falls in the long term.

**Figure 6.** A Rise in the Propensity to Consume Agricultural Goods

### A rise in the propensity to save

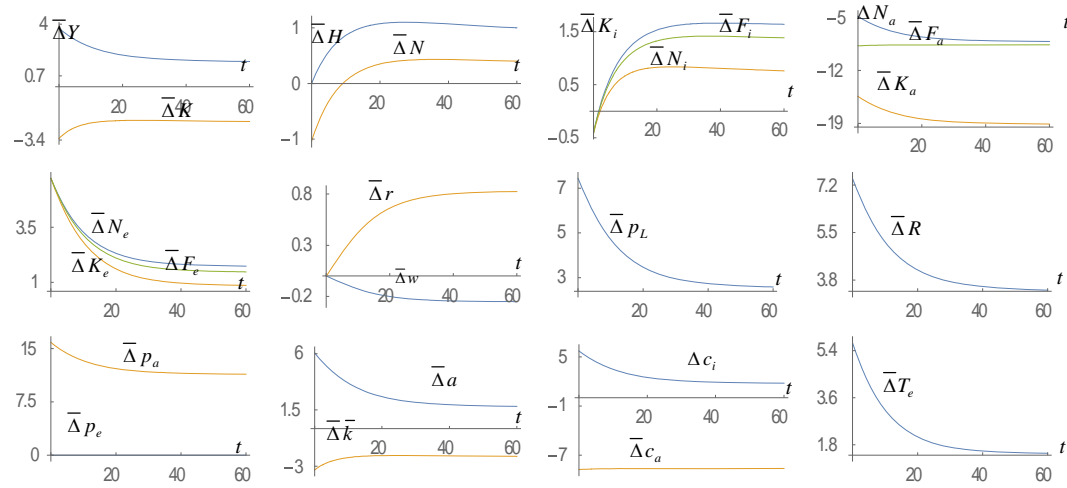
One of challenging questions is impact of changes in saving rates on economic structure and economic growth. Different economic theories argue for different effects. In Keynesian economic theory savings tend to reduced national income, while neoclassical growth theory tends to suggest the opposite effect. We now change the propensity to save as follows:  $\lambda_0: 0.8 \Rightarrow 0.81$ . As the household tends to save more out of the disposable income, the physical wealth is increased. The wage rate rises and the rate of interest falls. The total wealth falls initially and rises in the long term. The land value and land value falls initially and are enhanced in the long term. The price of education rises and the time spent on education falls initially and is slightly affected in the long term. The total labor supply is reduced. The human capital rises initially and falls in the long term. The labor inputs of the three sectors are reduced. The consumption levels of industrial and agricultural goods fall initially and are increased slightly. Hence, in the long term the household's consumption and wealth are enhanced. The GDP falls initially and rises in the long term. The output and capital input of the industrial sectors are expanded. The output and capital input of the agricultural sectors are reduced initially and expanded in the long term.

**Figure 7.** A Rise in the Propensity to Save**A rise in the output elasticity of land of the agricultural sector**

We now examine effects of the following rise in the output elasticity of the industrial sector:  $\alpha_a : 0.1 \Rightarrow 0.08$ ,  $\beta_a : 0.2 \Rightarrow 0.18$ . The specified changes imply that the output elasticity of land of the agricultural sector is increased.

$$\bar{\Delta} l_h = -1.6, \quad \bar{\Delta} L_a = 4.02.$$

A rise in the output elasticity of land of the agricultural sector implies that the land share of the sector's total revenue is increased in the optimal decision. Some land is shifted from housing to agriculture. The land value and land rent are enhanced. The price of agricultural goods is enhanced. The output and capital and labor inputs of the agricultural sector are reduced. The household spends more time on education and human capital is increased. The total labor supply falls initially and is augmented in the long term. The total capital stock is reduced and the GDP is increased. The output and capital and labor inputs of the agricultural sector are reduced initially and are increased in the long term. The consumption level of agricultural goods is reduced but the consumption level of industrial goods is augmented.

**Figure 8.** A Rise in the Output Elasticity of Land of the Agricultural Sector

## 5. Concluding remarks

This study examined dynamics of land value and land rent in a three-sector growth model with endogenous physical and human capital accumulation. We also emphasized endogenous economic structural change. The main framework is neoclassical and the household's decision is based on Zhang's alternative approach. The model is constructed by integrating some ideas in the neoclassical growth theory and land economics in a compact framework. By simulation, we demonstrated that the economic system has a unique steady state. The confirmed stability of the steady state implies that we can effectively conduct comparative dynamic analysis. We studied effects of changes in the propensity to receive education, the propensity to consume housing, the propensity to consume industrial goods, the propensity to consume agricultural goods, the propensity to save, and the output elasticity of land of the agricultural sector. We demonstrate that a rising period in the GDP tends to be in tandem with an increasing period in the land price (e.g., Liu *et al.*, 2011). It should be noted that many limitations of this model become apparent in the light of the sophistication of the literature of growth theory and land economics. For instance, we may generalize the model by using more general function forms of the two sectors and the utility function. It is also possible to extend the model by taking account of heterogeneity of households. We may introduce some kind of government intervention in education into the model. In this study, we don't consider public provision or subsidy of education. In the literature of education and economic growth, many models with heterogeneous households are proposed to address issues related to taxation, education policy, distribution of income and wealth, and economic growth (e.g., Bénabou, 2002; Glomm and Kaganovich, 2008; Nakajima and Nakamura, 2009).

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### Appendix: Proving the Lemma

The appendix shows that the dynamics can be expressed by one differential equation. From (3), (5) and (7), we obtain

$$z \equiv \frac{r + \delta_k}{w} = \frac{\tilde{\alpha}_i N_i}{K_i} = \frac{\tilde{\alpha}_a N_a}{K_a} = \frac{\tilde{\alpha}_e N_e}{K_e}, \quad (\text{A1})$$

where we omit time index and  $\tilde{\alpha}_j \equiv \alpha_j / \beta_j$ ,  $j = i, a, e$ . By (1) and (2), we have

$$r + \delta_k = \frac{\alpha_i A_i}{\tilde{\alpha}_i^{\beta_i}} z^{\beta_i}, \quad w = \frac{\tilde{\alpha}_i^{\alpha_i} \beta_i A_i}{z^{\alpha_i}}, \quad (\text{A2})$$

where we also use (A1). We express  $w$  and  $r$  as functions of  $z$ .

From (16) and (20), we get

$$\mu \bar{y} \bar{N} = p_a F_a. \quad (\text{A3})$$

From (5), we have

$$r + \delta_k = \frac{\alpha_a p_a F_a}{K_a}. \quad (\text{A4})$$

From (A4) and (4) we solve

$$p_a \left( \frac{L_a}{K_a} \right)^\zeta = \frac{\tilde{\alpha}_a^{\beta_a} (r + \delta_k)}{\alpha_a A_a z^{\beta_a}}, \quad (\text{A5})$$

where we use (A1). From (A3) and (A4), we solve

$$\mu \bar{y} \bar{N} = \left( \frac{r + \delta_k}{\alpha_a} \right) K_a. \quad (\text{A6})$$

By (5) and (A3), we have

$$R = \frac{\zeta \mu \bar{y} \bar{N}}{L_a}. \quad (\text{A7})$$

From  $R l_h = \eta \bar{y}$  in (16) and (A7), we have

$$\zeta \mu \bar{N} l_h = \eta L_a. \quad (\text{A8})$$

From (17) and (A8), we solve the land distribution as follows



$$L_a = \frac{\zeta \mu L}{\eta + \zeta \mu}, \quad l_h = \frac{\eta L}{(\eta + \zeta \mu) \bar{N}}. \quad (\text{A9})$$

The land distribution is invariant over time.

From the definition of  $\bar{y}$ , we have

$$\bar{y} = (1 + r) \bar{k} + \left(1 + \frac{1}{r}\right) R \bar{l} + H^m T_0 w, \quad (\text{A10})$$

where we use the definition of wealth and (8). From  $R l_h = \eta \bar{y}$  in (16) and (A10) we solve

$$R = \omega_1 \bar{k} + \omega_2, \quad (\text{A11})$$

where

$$\omega_1(z) \equiv \left\{ l_h - \left(1 + \frac{1}{r}\right) \eta \bar{l} \right\}^{-1} (1 + r) \eta, \quad \omega_2(z) \equiv \left\{ l_h - \left(1 + \frac{1}{r}\right) \eta \bar{l} \right\}^{-1} \eta T_0 H^m w.$$

From (A10) and (A11) we have

$$\bar{y} = \tilde{\omega}_1 \bar{k} + \tilde{\omega}_2, \quad (\text{A12})$$

where

$$\tilde{\omega}_1(z) \equiv (1 + r) + \left(1 + \frac{1}{r}\right) \bar{l} \omega_1, \quad \tilde{\omega}_2(z) \equiv \left(1 + \frac{1}{r}\right) \bar{l} \omega_2 + H^m T_0 w.$$

From (6) and (7) we have

$$p_e(z) = \frac{w z^{\alpha_e}}{\beta_e A_e \tilde{\alpha}_e^{\alpha_e}}. \quad (\text{A13})$$

where we also use (A1). From the definition of  $\bar{p}_e$  we have

$$\bar{p}_e(z, H) = p_e(z) + H^m w. \quad (\text{A14})$$

From (16) and (A12) we have

$$T_e = \frac{\kappa \tilde{\omega}_1}{\bar{p}_e} \bar{k} + \frac{\kappa \tilde{\omega}_2}{\bar{p}_e}. \quad (\text{A15})$$

From (A15) and (1) we get

$$N = p_1 - p_2 \bar{k}, \quad (\text{A16})$$

where we also use the time constraint and

$$p_1(z, H) \equiv \left( T_0 - \frac{\kappa \tilde{\omega}_2}{\bar{p}_e} \right) H^m \bar{N}, \quad p_2(z, H) \equiv \frac{\kappa \tilde{\omega}_1 H^m \bar{N}}{\bar{p}_e}.$$

Insert (A1) in  $N_i + N_a + N_e = N$

$$\frac{K_i}{\tilde{\alpha}_i} + \frac{K_a}{\tilde{\alpha}_a} + \frac{K_e}{\tilde{\alpha}_e} = \frac{N}{z}. \quad (\text{A17})$$

From (22) and (23) we have

$$K_i + K_a + K_e = \bar{k} \bar{N}. \quad (\text{A18})$$

Insert (A12) and (A6) we solve

$$K_a = \hat{\omega}_1 \bar{k} + \hat{\omega}_2, \quad (\text{A19})$$

where

$$\hat{\omega}_1(z, H) \equiv \left( \frac{\alpha_a}{r + \delta_k} \right) \mu \bar{N} \tilde{\omega}_1, \quad \hat{\omega}_2(z, H) \equiv \left( \frac{\alpha_a}{r + \delta_k} \right) \mu \bar{N} \tilde{\omega}_2.$$

Insert (A19) and (A16) in (A17) and (A18)

$$\frac{K_i}{\tilde{\alpha}_i} + \frac{K_e}{\tilde{\alpha}_e} = b_1(z, H) \equiv \bar{p}_1 - \bar{p}_2 \bar{k}, \quad (\text{A20})$$

$$K_i + K_e = b_2(z, H) \equiv \bar{p}_0 \bar{k} - \hat{\omega}_2,$$

where

$$\bar{p}_1(z, H) \equiv \frac{p_1}{z} - \frac{\hat{\omega}_2}{\tilde{\alpha}_a}, \quad \bar{p}_2(z, H) \equiv \frac{p_2}{z} + \frac{\hat{\omega}_1}{\tilde{\alpha}_a}, \quad \bar{p}_0(z, H) \equiv \bar{N} - \hat{\omega}_1.$$

Solve (A20)

$$K_i = \alpha_0 \left( b_1 - \frac{b_2}{\tilde{\alpha}_e} \right), \quad K_e = \alpha_0 \left( \frac{b_2}{\tilde{\alpha}_i} - b_1 \right), \quad (\text{A21})$$

where

$$\alpha_0 = \left( \frac{1}{\tilde{\alpha}_i} - \frac{1}{\tilde{\alpha}_e} \right)^{-1}.$$

Insert the definitions of  $b_j$  in (A20) in (A21)

$$K_i = m_i \bar{k} + \bar{m}_i, \quad K_e = m_e \bar{k} - \bar{m}_e, \quad (\text{A22})$$

where

$$m_i(z, H) \equiv -\alpha_0 \left( \bar{p}_2 + \frac{\bar{p}_0}{\tilde{\alpha}_e} \right), \quad \bar{m}_i(z, H) \equiv \alpha_0 \left( \bar{p}_1 + \frac{\hat{\omega}_2}{\tilde{\alpha}_e} \right),$$

$$m_e(z, H) \equiv \alpha_0 \left( \frac{\bar{p}_0}{\tilde{\alpha}_i} + \bar{p}_2 \right), \quad \bar{m}_e(z, H) \equiv \alpha_0 \left( \frac{\hat{\omega}_2}{\tilde{\alpha}_i} + \bar{p}_1 \right).$$

We solved the capital distribution as functions of  $z$ ,  $H$  and  $\bar{k}$ . By (A1), we solve the labour distribution as functions of  $z$ ,  $H$  and  $\bar{k}$  as follows

$$N_i = \frac{z K_i}{\tilde{\alpha}_i}, \quad N_a = \frac{z K_a}{\tilde{\alpha}_a}, \quad N_e = \frac{z K_e}{\tilde{\alpha}_e}. \quad (\text{A23})$$

From (6) and (19) we have

$$T_e \bar{N} = K_e A_e \left( \frac{z}{\tilde{\alpha}_e} \right)^{\beta_e}, \quad (\text{A24})$$

where we also use (A1). Insert (A22) and (A15) in (A24)

$$\bar{k}(z, H) = \left[ \frac{\kappa \tilde{\omega}_2 \bar{N}}{\bar{p}_e} + \left( \frac{z}{\tilde{\alpha}_e} \right)^{\beta_e} A_e \bar{m}_e \right] \left[ \left( \frac{z}{\tilde{\alpha}_e} \right)^{\beta_e} A_e m_e - \frac{\kappa \tilde{\omega}_1 \bar{N}}{\bar{p}_e} \right]^{-1}. \quad (\text{A25})$$

From (8) and (9) we have

$$a = \phi(z, H) \equiv \bar{k} + \frac{\bar{l} R}{r}. \quad (\text{A26})$$

It is straightforward to check that all the variables can be expressed as functions of  $z$  and  $H$  at any point in time as follows:  $r$  and  $w$  by (A2)  $\rightarrow \bar{k}$  by (A25)  $\rightarrow K_a$  by (A19)  $\rightarrow K_i$  and  $K_e$  by (A22)  $\rightarrow N_i, N_e$ , and  $N_a$  by (A1)  $\rightarrow N$  by (A16)  $\rightarrow p_e$  by (A13)  $\rightarrow \bar{y}$  by (A13)  $\rightarrow R$  by (A11)  $\rightarrow p_L$  by (A13)  $\rightarrow a$  by (A26)  $\rightarrow L_a$  and  $l_h$  by (A9)  $\rightarrow p_a$  by (A5)  $\rightarrow T_e, c_i, c_a$ , and  $s$  by (16)  $\rightarrow T$  by (14)  $\rightarrow C_a$  by (20)  $\rightarrow C_i = c_i \bar{N} \rightarrow p_L$  by (A23)  $\rightarrow p_a$  by (A5)  $\rightarrow F_i$  by (2)  $\rightarrow F_a$  by (4)  $\rightarrow F_e$  by (6). From this procedure, (17) and (18), we have

$$\dot{a} = \Lambda_0(z, H) \equiv s - a, \quad (\text{A27})$$

$$\dot{H}(t) = \Omega(z, H) \equiv \frac{\nu_e F_e^{a_e} (H^m T_e \bar{N})^{\beta_e}}{H^{\pi_e} \bar{N}} + \frac{\nu F_i^{a_i}}{H^{\pi_i} \bar{N}} + \frac{\nu_h C_i^{a_h}}{H^{\pi_h} \bar{N}} - \delta_h H. \quad (\text{A28})$$

Taking derivatives of (A26) with respect to  $t$  yields

$$\dot{a} = \frac{\partial \phi}{\partial z} \dot{z} + \Omega \frac{\partial \phi}{\partial H}, \quad (\text{A29})$$

in which we use (A28). Equal (A27) and (A29)

$$\dot{z} = \Lambda(z, H) \equiv \left( \Lambda_0 - \Omega \frac{\partial \phi}{\partial H} \right) \left( \frac{\partial \phi}{\partial z} \right)^{-1}. \quad (\text{A30})$$

From (A28) and (A30), we determine the motion of  $z$  and  $H$ . We thus proved the lemma.