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Mathematical model used in substantiating optimal contract

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Abstract. *The theory of optimal contracts refers to the market state, in which the bidders, the participants in the execution of transactions, have a certain number of certain information higher or lower. Normally, anyone who concludes a contract, regardless of its nature and we refer to the business environment, seeks to place himself on an optimal solution, i.e. one that according to the Latin principle "aurea mediocritas" gives him a chance to average, but protects him the realization of a contract subject to many risks. Normally, the conclusion of any contract is based on an interest, which starting from the principle of the free market, based on the ratio between supply and demand, may end up in the situation of concluding one or another of the contracts.*

In the literature the problem of optimal contracts is not so new, only that there is less information and materials published by specialists on this topic. In principle, in the capital market, those operating in the capital market must consider the possibility of concluding a contract on optimal terms.

Nowadays, when we are in the Big Data era, in which databases are enormous, it is essential that companies or agencies that mediate business know very clearly the information that underlies the transaction to be concluded. As always, there is a clear enough difference between the level of information held by one or another of the customers.

In this article we started the theoretical problem in very synthetic terms, because it is known and we tried to substantiate a model that could be the basis for renting optimal contracts. It should be noted that it started from the utility function, in the sense of von Neumann-Morgenstern, as well as the Lagrange function and last but not least from the Kuhn-Tucker multiplier, often used in microeconomic analyzes based on consistent models.

In this article we started from the estimation of the multiplier between two equations to obtain an optimal result. The optimal contract for an agent, best placed, is a solution given by a system of two

equations that lead to Pareto optimality, or if you will to Pareto efficiency. On the other hand, the optimal contract in the situation of asymmetric information for the interested agent is the optimal Pareto or Pareto-optimal. In this situation, in this article we took a numerical case, from whose analysis it is clear how the study should be based in the perspective of substantiating the decisions to conclude an optimal contract.
Finally, the mathematical model is concretely formulated, imposing some participation restrictions or compatibility restrictions that must always be taken into account by the one who enters the market and wants to conclude an optimal contract. It follows that the model can be simplified, in order to remove some restrictions or to establish that some of them can be satisfied by using the Kuhn-Tucker multiplier. Thus, if a second agent or another market participant signs the contract, he assumes some risks. There are a lot of solutions in the market study, however, and then it must be borne in mind that these contracts are satisfactory if we take into account the conditions of the mathematical model we talked about.
In this article we started from the theoretical elements, analyzing them in the mathematical-economic sense, in order to reach the relationship that the conditions for concluding an optimal contract imply.

Keywords: optimal contract, models, functions, multiplier, agents, capital market.

JEL Classification: C02, E22, G11.

Introduction

In this article, the authors started from the essence offered by the free market, in the sense that any transaction must be based on a certain substrate of anticipated efficiency of profitability for the one who enters into this contract. Discussing these market elements, we followed starting from the Kuhn-Tucker method of nonlinear optimization, as well as from the von Neumann-Morgenstern utility function and we arrived through the mathematical logic at establishing an optimal contract.

In the article on the λ multiplier we issued some opinions, materialized in proven sentences, which lead to the fact that this optimal contract in a symmetric information situation for an agent is of the Pareto-optimal type. From here we developed the mathematical model, represented and graphically to be more suggestive, so as to reach a precise relationship that is based on how the market should be studied, using the proposed mathematical model.

We also took two numerical examples that we introduced and subjected to calculations and analyzes, examples that give the possibility to establish that the model used has certain limits and through the demonstration we provided those interested the opportunity to use it when they want to substantiate an optimal contract. Of course, the topic is very broad, but we tried to synthesize it as much as possible, so that mathematical relationships, deriving from each other based on certain logical criteria, lead us to the end of an efficient and usable mathematical model. Of course, this mathematical model can be used not only in negotiating and concluding optimal contracts, but also to establish optimal correlations and proportions at the macroeconomic level. It is simple to just replace the terms of this model with the ones we are interested in and the results will be satisfactory. We also mention this in the sense that a macroeconomic forecast must always be based on macroeconomic correlations and proportions that ensure macrostability and in this way the plans of measures or strategies of national economic development are in close accordance with the realities that may materialize at some point.

In the article, we have carefully tried to ensure that the notations used are compliant, consistent and easily identified through subsequent transformations based on the offer that the literature makes available to those interested.

The article also used some graphical representations precisely to solve and give logical meaning to the interpretation of the successive stages and led to the substantiation of this mathematical model that can be used mainly in optimal contracts.

Literature review

Anghelache Constantin, Anghel Mădălina Gabriela (2018, 2019), they address in their works various problems related to economic modeling and statistical-econometric analyzes applied to various economic phenomena. Anghelache Constantin (2009), addresses some methods and models for measuring financial-banking risks. Also, Anghelache Constantin (2019) presents in his paper some considerations on the effect of financial-monetary measures on the business environment. Cotfas Mihai, Andrei Anca, Roman Mihai, Marin Dumitru, Stancu Stelian, Patrașcu Nicolae (1995) addresses in their work problems related to the theory of general equilibrium. Marin Dumitru, Galupa Angela (2001), addresses theoretical issues related to externalities and public goods. Marin Dumitru, Manole Sorin, Turmacu Mihaela (2004) are concerned with the applicability of mathematics in economics. Marin Dumitru, Emanuel Lazăr, Pătrașcu Neculai (1998), addresses the theoretical aspects of capital market contracts. Marinescu Daniela, Marin Dumitru (2011a) addresses the analysis of certain economic phenomena at the microeconomic level. Marinescu Daniela, Marin Dumitru (2011b) analyzes an adverse selection model with three states of nature, in which both the director and the agent are risk neutral.

Methodologies, data, results and discussions

Adverse selection patterns occur when one of the two participants in *the principal-agent* model, namely *the principal (the decision-maker)*, has less information than the *agent*. In other words, the principal does not know the type of *agent*.

A trader does not know how big the buyer's budget is or an insurance company (as a decision maker) does not know how carefully the one who wants to insure drives, etc.

In [1] it is shown that the optimal for the principal (the one proposing the contract) is to propose a menu of contracts one for each type of agent and to build the model in such a way that each *agent* chooses his intended contract.

We will first analyze the case with two types of agents and how the reward changes in relation to the type of *agent*, in a situation of symmetrical information. If an agent chooses the action which from a lot of given shares we will assume that the gross profit of *the principal* is *P(a),* where $P'(a) > 0$ and $P''(a) < 0$.

Then the net profit is:

$$P(a) - s, \tag{1}$$

where *s* is represented the salary (reward) or transfer of the principal to the *agent.*

We assume that the principal recognizes the two types of agents denoted 1 and 2 as it is better placed or not.

The difference between the two agents is materialized depending on the utility. For the first *agent* the utility function is:

$$U(s^1, a^1) = U(s^1) - V(a^1) \tag{2}$$

where $U(\cdot)$ represents the utility function in the von Neumann-Morgenstern sense (in relation to gain s^1);

$V(a^1)$ represents the cost function of the effort following the agent's decision to choose the action $\hat{a}$.

In general the function $U(\cdot)$ has the properties: $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ (that is, the agent is either risk aversion or risk neutral). The cost function of the effort $V(\cdot)$ has the usual properties namely: $V'(\cdot) > 0$ and $V''(\cdot) \geq 0$.

The greater the effort, the greater the optimality of the agent. The optimal contract for the type 1 agent is determined by solving the nonlinear optimization program (using the Kuhn-Tucker method).

$$\underset{a^1,s^1}{\text{Max}}[P(a^1) - s^1] \quad (3)$$

s.r. $U(s^1) - V(a^1) \geq \underline{U}$ (the condition of participation in the contract)

$s^1 \geq 0, a^1 \geq 0$

where $\underline{U}$ is the minimum market threshold reserved for Agent 1.

The associated Lagrange function is:

$$L(s^1, a^1, \lambda) = P(a^1) - s^1 + \lambda[U(s^1) - V(a^1) - \underline{U}] \quad (4)$$

where $\lambda \geq 0$ is the Kuhn-Tucker multiplier.

The first order conditions that are both necessary and sufficient (given the properties of the *U* and *V*) become:

$$\frac{\partial L}{\partial s^1} \leq 0,\ s^1 \geq 0 \text{ and } s^1 \cdot \frac{\partial L}{\partial s^1} = 0 \quad (5)$$

$$\frac{\partial L}{\partial a^1} \leq 0,\ a^1 \geq 0 \text{ and } a^1 \cdot \frac{\partial L}{\partial a^1} = 0 \quad (6)$$

$$\frac{\partial L}{\partial \lambda} \leq 0,\ \lambda \geq 0 \text{ and } \lambda \cdot \frac{\partial L}{\partial \lambda} = 0 \quad (7)$$

or rewritten:

$$-1 + \lambda U'(s^1) \leq 0, s^1 \geq 0 \ \text{și}\ s^1[-1 + \lambda U'(s^1)] = 0 \quad (8)$$

$$P'(a^1) + \lambda V'(a^1) \leq 0, a^1 \geq 0 \ \text{și}\ a[P'(a^1) + \lambda V'(a^1)] = 0 \quad (9)$$

It follows from relation (5) that $\lambda > 0$, otherwise $P'(a^1) \leq 0$ contrary to the assumption of the function of profit.

Then the group of relations (3) is equivalent to $\frac{\partial L}{\partial \lambda} = 0$, well:

$$U(s^1) - V(a^1) - \underline{U} = 0 \text{ sau } U(s^1) - V(a^1) = \underline{U} \quad (10)$$

that is, the participation restriction is saturated. If the agent accepts the contract, he receives exactly the minimum threshold reserved by the market as a utility level.

It follows from relation (6) $s^1 > 0$, and from (4) we have the equation $-1 + \lambda U'(s^1) = 0$, and from (5) $P'(a^1) + \lambda V'(a^1) = 0$

We eliminate the multiplier λ between the last two equations and obtain the following result:

Proposition 1: The optimal contract for agent 1 (better placed) is the solution of the system given by the following two equations:

$$\begin{cases} P'(a^1) = \frac{V'(a^1)}{U'(s^1)} \\ U(s^1) - V(a^1) = \underline{U} \end{cases} \tag{11}$$

The first equation characterizes the Pareto optimality (efficiency) of the contract, i.e. the gross marginal profit equals the marginal rate of substitution of the reward by the level of action (the level of effort made).

The second equation shows the maximum level of satisfaction obtained by the *agent*.

For the second type of *agent*, the satisfaction function is given by the relationship:

$$U(s^2) - (k + f(\varepsilon))V(a^2) \geq \underline{U} \tag{12}$$

where it is fixed $k > 1$, and $f(\varepsilon)$ is a strictly ascending function with $f(0) = 0$,

and so$f(\varepsilon) \in (1 - k, \infty)$.

We also assume that $f(\cdot)$it is derivable, therefore $f'(\varepsilon) > 0$

The mathematical model, in this case, is written:

$$\underset{a^2,s^2}{\text{Max}}[P(a^2) - s^2]$$

$$\text{s.r. } U(s^2) - [k + f(\varepsilon)]V(a^2) \geq \underline{U},\ \ s^2 \geq 0,\ \ a^2 \geq 0 \tag{13}$$

Applying the Kuhn-Tucker method again, we obtain the following system:

$$\begin{cases} P'(s^2) = \frac{[k+f(\varepsilon)]V(a^2)}{U'(s^2)} \\ U'(s^2) - [k + f(\varepsilon)]V(a^2) = \underline{U} \end{cases} \tag{14}$$

Coefficient $k + f(\varepsilon)$ fneedles the difference between the two types of agents namely: at the same level of action *a*, the usefulness of agent 2 is greater than that of agent 1.

From the second equation we get the reward $\tilde{s}^2$ for a level of effort $\tilde{a}^2$, according to the relationship:

$$\tilde{s}^2 = U^{-1}\{[k + f(\varepsilon)]V(\tilde{a}^2) + \underline{U}\} \tag{15}$$

Proposition 2: The optimal contract in symmetric information situation for the type 2 agent is PARETO-optimal.

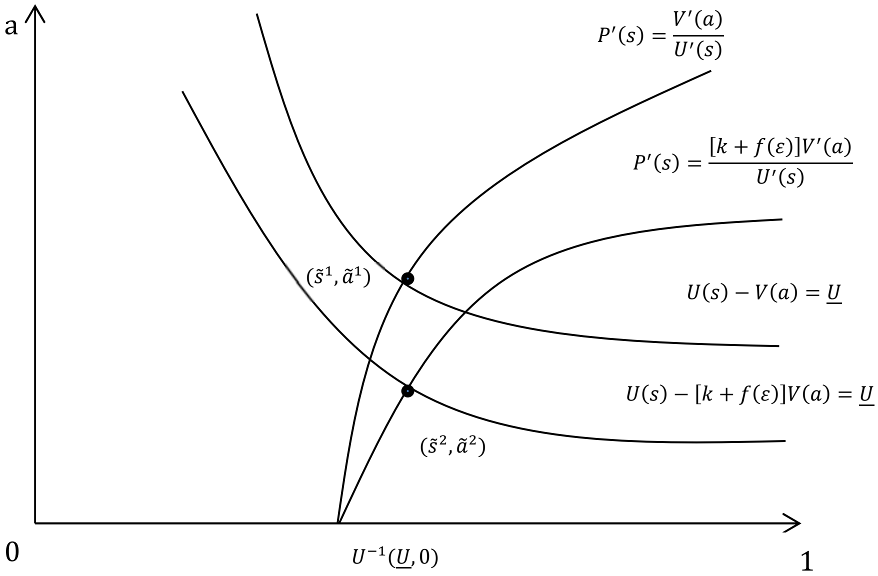
The reward depends on the parameter *ε* and is given by the formula:

$$\frac{\partial s^1}{\partial \varepsilon} = U^{-1}\{[k + f(\varepsilon)]V(\tilde{a}^2) + \underline{U}\} \tag{16}$$

It turns out that as the difference between the two types increases and the reward for the better placed type increases.

A graphical analysis of the two contracts will allow us to compare the two levels of action $\tilde{a}^1$ and $\tilde{a}^2$, such as rank I rewards.

Graph 1. *Evolution of the two contracts*



Curves $U(s) - V(a) = \underline{U}$ and $P'(s) = \frac{V'(a)}{U'(s)}$ characterizes the optimal contract for the type 1 agent, and the equilibrium point is $(\tilde{s}^1, \tilde{a}^1)$.

For type 2 agent, i.e. curves $P'(s) = \frac{[k+f(\varepsilon)]V'(a)}{U'(s)}$ and $U(s) - [k + f(\varepsilon)]V(a) = \underline{U}$, the optimal contract is on point $(\tilde{s}^2, \tilde{a}^2)$.

Is it necessary to justify the position of the four curves?

The graph shows that $\tilde{s}^1 > \tilde{s}^2$, in other words, the better placed agent will be required to make a greater effort.

Regarding the rewards, nothing can be said about the relationship between them. There are two effects, namely: the type 1 agent, exerting a greater effort, the principal will transfer a larger amount. On the other hand, the type 2 agent, due to the coefficient $k + f(\varepsilon) > 1$, it will have a higher cost-effectiveness (disuse) function and will consequently claim a higher reward.

The composition of the two opposite effects will also result in the relationship (inequality) between the two rewards. The higher the parameter ε, the further away from the type 2 agent.

Particular case:

Whether $(a) = 600a$, $U(s) = \sqrt{s}$ and $V(a) = a^2$, and $f(\varepsilon) = 10 \cdot \varepsilon + 1$. We assume that $\underline{U} = 0$ and $k = 2$. For agent 1, the optimal contract results from solving the system:

$$\begin{cases} P'(a) = \frac{V'(a)}{U'(s)} \\ U(s) - V(a) = \underline{U} \end{cases}, \text{ so } 600 = \frac{2a}{\frac{1}{2\sqrt{s}}} \Rightarrow 600 = 4a\sqrt{s}$$

The second equation becomes $\sqrt{s} = a^2$ and so $600 = 4a^3 => a^3 = 125$, $\tilde{a}^1 = 5$

Then $\tilde{s}^1 = 625$

For agent 2, we get:

$$\begin{cases} 600 = \frac{(3+\varepsilon)\cdot 2a}{\frac{1}{2\sqrt{s}}} \\ \sqrt{s} - (3+\varepsilon)\cdot a^2 = 0 \end{cases} \text{ or } \begin{cases} 600 = 4\cdot(3+\varepsilon)\cdot a\sqrt{s} \\ \sqrt{s} = (3+\varepsilon)\cdot a^2 \end{cases} =>$$

$=> 125 = (3+\varepsilon)^2 \cdot a^3$,

where from $\tilde{a}^2 = \frac{5}{\sqrt[3]{(3+\varepsilon)^2}}$,

and $\tilde{s}^2 = (3+\varepsilon)^2 \cdot (\tilde{a}^2)^4 = (3+\varepsilon)^2 \cdot \frac{5}{\sqrt[3]{(3+\varepsilon)^2}} = 5\cdot(3+\varepsilon)^{8/3}$.

It is observed that as ε increases, the second agent (the inefficient one) will be required to exert less effort.

Next we will analyze the P-A model in an asymmetric information situation, respectively if *the principal* does not recognize the type of agent.

We will assume, however, that P believes that A is type 1 (efficient) with a probability *p*. These probabilities can be established on objective basis, possibly using the past experience of economic or subjective agents (Bayesian type). The decision maker will in this case propose two contracts, one for each agent and will build the mathematical model in such a way that each agent chooses his intended contract (if the contract is signed).

The mathematical model will contain the objective function which consists in maximizing the net profit expected for the decision maker and the actual restrictions. These are of two kinds:

- Participation restrictions, one for each agent;
- Incentive compatibility restrictions (with the type of agent), i.e. the restrictions by which each agent is encouraged to choose (if he signs) even his intended contract.

Whether (a^1, s^1) and respectively (a^2, s^2) contracts for the two agents, efficiently and effectively. The difference between the two types of agents is given by the cost function of the effort, namely:

i) for the agent (efficient) 1, $V(a)$ with the usual properties: $V'(a) \geq o$ and $V''(a) \geq 0$;
ii) for the agent (inefficient) 2, $[k + f(\varepsilon)]V(a)$ where *k* is a fixed parameter$k > 1$, and the function $f: [0, \infty) \rightarrow (1-k, \infty)$ it is derivable and strictly ascending.

Given the above, the mathematical model is thus formalized:

I) $\max_{\substack{(a^1,s^1)\\(a^2,s^2)}} \{p[P(a^1) - s^1] + (1-p)[P(a^2) - s^2]\}$ (17)

s.r. II) $U(s^1) - V(a^1) \geq \underline{U}$ (18)

$U(s^2) - [k + f(\varepsilon)]V(a^2) \geq \underline{U}$ (19)

$$U(s^1) - V(a^1) \geq U(s^2) - V(a^2) \quad (20)$$

$$U(s^2) - [k + f(\varepsilon)]V(a^2) \geq U(s^1) - [k + f(\varepsilon)]V(a^1) \quad (21)$$

III) a^1, s^1, a^2, s^2 positive

Restrictions (18) and (19) are participation restrictions, and (20) and (21) are incentive compatibility restrictions by which agents are motivated to choose their intended contract.

The model can be simplified in order to remove some restrictions or to establish that some restrictions (in case of admissible solutions) can be satisfied or not with implications on the values of Kuhn-Tucker multipliers.

Proposition 1: If restrictions (19) and (20) are verified, then restriction (18) is also verified.

Demonstration: In other words, if the second agent, the worst placed, signs the contract, the more agent 1 signs the contract.

Indeed from (20) and taking into account the range of values of the function $f(\varepsilon)$, get:

$$U(s^1) - V(a^1) \geq U(s^2) - V(a^2) \geq U(s^2) - [k + f(\varepsilon)]V(a^2) \geq \underline{U}$$

So $U(s^1) - V(a^1) \geq \underline{U}$, that is, the constraint (18) is redundant and can be omitted.

Proposition 2: If the set of admissible solutions is empty, then $a^1 \geq a^2$

Demonstration: Indeed, assuming that there is at least one pair of contracts (a^1, s^1), (a^2, s^2), one for each participant and adding the restrictions (20) and (21), get:

$$-V(a^1) - [k + f(\varepsilon)]V(a^2) \geq -V(a^2) - [k + f(\varepsilon)]V(a^1)$$

(after a convenient reduction in the useful life of the reward)

We group and we get:

$$[V(a^1) - V(a^2)][1 - k - f(\varepsilon)] \leq 0 \quad (22)$$

How $f(\varepsilon) > 1 - k$, results from (22), $V(a^1) - V(a^2) \geq 0$,and from the monotony of function $V(\cdot), a^1 \geq a^2$ the condition of enforceability of the contract requires that the better placed (efficient) agent be required to submit a higher level of action.

We will solve the problem using the Kuhn-Tucker method and consequently rewrite the remaining constraints, respectively (19), (20) and (21) after removing the constraint (18), and the Kuhn-Tucker multipliers $\lambda_1, \lambda_2, \lambda_3$ they are positive.

The Lagrange function is written:

$$L(a^1, s^1, a^2, s^2, \lambda_1, \lambda_2, \lambda_3) = p[P(a^1) - s^1] + (1 - p)[P(a^2) - s^2] +$$

$$+ \lambda_1\{U(s^2) - [k + f(\varepsilon)]V(a^2) - \underline{U}\} + \lambda_2[U(s^1) - V(a^1) - U(s^2) + V(a^2)] +$$

$$+\lambda_3\{U(s^2) - [k + f(\varepsilon)]V(a^2) - U(s^1) - [k + f(\varepsilon)]V(a^1)\} \quad (23)$$

We will analyze, without restricting the generality of the problem, an optimal interior (with all the strictly positive components).

Then from the relationship group $\frac{\partial L}{\partial x} \leq 0,\ x \geq 0$ and $x \cdot \frac{\partial L}{\partial x} = 0$, where $x \in \{a^1, s^1, a^2, s^2\}$ we get, if $x > 0, \frac{\partial L}{\partial x} = 0$. For $x = a^1$ results $\frac{\partial L}{\partial a^1} = 0$, where from:

$pP'(a^1) - \lambda_2 V'(a^1) + \lambda_3[k + f(\varepsilon)]V'(a^1) = 0$

or

$$\frac{pP'(a^1)}{V'(a^1)} = \lambda_2 - \lambda_3[k + f(\varepsilon)]. \qquad (24)$$

Analogous to the other variables:

$\frac{\partial L}{\partial s^1} = 0$, deci $- p + \lambda_2 U'(s^1) - \lambda_3 U'(s^1) = 0$, so:

$$\frac{p}{U'(s^1)} = \lambda_2 - \lambda_3 \qquad (25)$$

$\frac{\partial L}{\partial a^2} = 0, (1-p)P'(a^2) - \lambda_1[k + f(\varepsilon)]V'(a^2) + \lambda_2 V'(a^2) - \lambda_3[k + f(\varepsilon)]V'(a^2) = 0$
where from:

$$(1-p)\frac{P'(a^2)}{V'(a^2)} = \lambda_1[k + f(\varepsilon)] - \lambda_2 + \lambda_3[k + f(\varepsilon)] \qquad (26)$$

$\frac{\partial L}{\partial s^2} = 0, -(1-p) + \lambda_1 U'(s^2) - \lambda_2 U'(s^2) + \lambda_3 U'(s^2) = 0$

$$\frac{(1-p)}{U'(s^2)} = \lambda_1 - \lambda_2 + \lambda_3. \qquad (27)$$

From relations (24) and (26), respectively (25) and (27) by assembly we obtain:

$$\frac{pP'(a^1)}{V'(a^1)} + (1-p)\frac{P'(a^2)}{V'(a^2)} = \lambda_1[k + f(\varepsilon)] \qquad (28)$$

and

$$\frac{p}{U'(s^1)} + \frac{1-p}{U'(s^2)} = \lambda_1. \qquad (29)$$

From both relations it follows that the multiplier λ_1 it is strictly positive.

From the first-order Kuhn-Tucker conditions it follows that $\frac{\partial L}{\partial \lambda_1} = 0$, well:

$U(s^2) - [k + f(\varepsilon)]V(a^2) - \underline{U} = 0$

At the optimum point, agent 2 obtains exactly the minimum threshold reserved by the market, i.e.:

$$U(s^2) - [k + f(\varepsilon)]V(a^2) = \underline{U} \qquad (30)$$

The higher the parameter ε, the greater the difference between the two types and at the same time the inefficiency of the inefficient agent.

Analyzing equation (25) and taking into account the property of the function $U(\cdot)$ it turns out that $\lambda_2 > 0$ and $\frac{\partial L}{\partial \lambda_2} = 0$, well:

$$U(s^1) - V(a^1) = U(s^2) - V(a^2) \tag{31}$$

By processing this equation we obtain the second characteristic of the contract:

$$U(s^1) - V(a^1) = U(s^2) - [k + f(\varepsilon)]V(a^2) + [k + f(\varepsilon)]V(a^2) - V(a^2) =$$

$$= \underline{U} + [k - 1 + f(\varepsilon)]V(a^2) \tag{32}$$

The term $[k - 1 + f(\varepsilon)]V(a^2)$, also called conformational rent, it represents the utility increase obtained by agent 1 due to the fact that it is better placed.

I showed above that $a^1 \geq a^2$ (Proposition 2)

Proposition 3: At the optimum point $a^1 > a^2$

Demonstration: We assume the opposite, namely that $a^1 = a^2 = a$. Then it follows from the equality (31) that and $s^1 = s^2 = s$. Relationships (24), (25), (26) and (27) become:

$$\frac{pP'(a)}{V'(a)} = \lambda_2 - \lambda_3[k + f(\varepsilon)]$$

$$\frac{(1-p)P'(a)}{V'(a)} = \lambda_1[k + f(\varepsilon)] - \lambda_2 + \lambda_3[k + f(\varepsilon)] \tag{33}$$

and

$$\frac{p}{U'(s)} = \lambda_2 - \lambda_3$$

$$\frac{1-p}{U'(s)} = \lambda_1 - \lambda_2 + \lambda_3 \tag{34}$$

Eliminating multipliers λ_2 and λ_3 it turns out that $\lambda_2 = 0$, in contradiction with (25). So $a^1 > a^2$. Using this inequality and the restrictions (10) and (11) we obtain $\lambda_3 = 0$. Then from (24) and (25) we obtain the third characteristic of the optimal contract, namely:

$$P'(a^1) = \frac{V'(a^1)}{U'(s^1)} \tag{35}$$

The optimal contract for agent 1 is also Pareto-optimal.

Application

Either the particular case where $P(a) = 10a$, $p = \frac{1}{2}$, $U(s) = s$, $V(a) = (a)^2$, $\underline{U} = 0$, $k = 2$ and $f(\varepsilon) = 4\varepsilon + 1$, $\varepsilon > -\frac{1}{2}$.

The optimal contract is determined as a solution of the nonlinear optimization program:

$$\max_{\substack{s^1,a^1\\ s^2,a^2}} \left[\frac{1}{2}(10a^1 - s^1) + \frac{1}{2}(10a^2 - s^2)\right]$$

s.r. $s^1 - (a^1)^2 \geq 0$

$s^2 - (3 + 4\varepsilon)(a^2)^2 \geq 0$

$s^1 - (a^1)^2 \geq s^2 - (a^2)^2$

$s^2 - (3 + 4\varepsilon)(a^2)^2 \geq s^1 - (3 + 4\varepsilon)(a^1)^2$

$s^1 \geq 0, a^1 \geq 0, s^2 \geq 0, a^2 \geq 0$

We note with: $R^1 = s^1 - (a^1)^2$ și $R^2 = s^2 - (3 + 4\varepsilon)(a^2)^2$

Then the mathematical model becomes:

$$\underset{\substack{R^1,a^1\\R^2,a^2}}{\text{Max}} \left\{\frac{1}{2}[(10a^1 - (a^1)^2] + \frac{1}{2}[10a^2 - (3 + 4\varepsilon)(a^2)^2] + \left[\frac{1}{2}R^1 + \frac{1}{2}R^2\right]\right\}$$

The first two restrictions are reduced to sign conditions $R^1 \geq 0,\ R^2 \geq 0$, and the last two restrictions become:

$R^1 \geq R^2 + 2(1 + 2\varepsilon)(a^2)^2$

and

$R^2 \geq R^1 - 2(1 + 2\varepsilon)(a^2)^2$

It is known that at the optimum point $R^1 = R^2 + 2(1 + 2\varepsilon)(a^2)^2$, and $R^2 = 0$.

The problem comes down to optimizing a function of two variables, namely:

$$\underset{a^1,a^2}{\text{Max}}\, F(a^1, a^2)$$

where: $F(a^1, a^2) = \frac{1}{2}[(10a^1 - (a^1)^2] + \frac{1}{2}[10a^2 - (3 + 4\varepsilon)(a^2)^2] - (1 + 2\varepsilon)(a^2)^2$

The first order conditions are:

$\frac{\partial L}{\partial a^1} = 0 \Rightarrow 10 - 2a^1 = 0,$ where from $\tilde{a}^1 = 5$

$\frac{\partial L}{\partial a^2} = 0 \Rightarrow \frac{1}{2}[10 - 2(3 + 4\varepsilon)a^2] - 2(1 + 2\varepsilon)a^2 = 0 \Rightarrow 5 - (3 + 4\varepsilon)a^2 - 2(1 + 2\varepsilon)a^2 = 0,$ where from $\tilde{a}^2 = \frac{5}{5+8\varepsilon}$

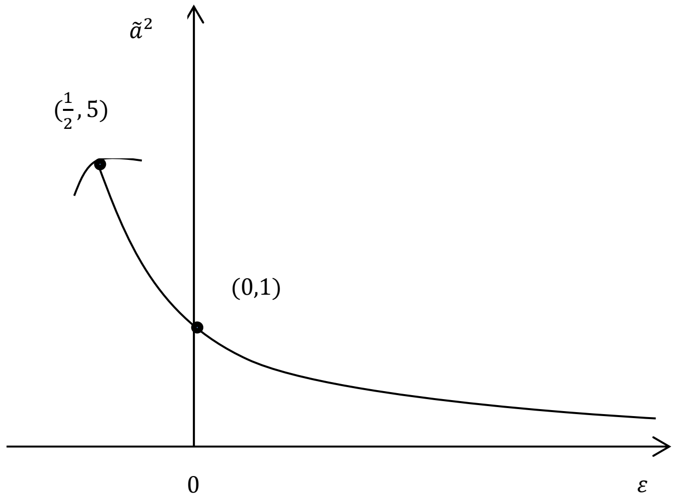
The optimal contract is written:

$\tilde{a}^1 = 5,\ \tilde{s}^1 = 25 + \frac{50(1+2\varepsilon)}{(5+8\varepsilon)^2}$

$\tilde{a}^2 = \frac{5}{5+8\varepsilon},\ \tilde{s}^2 = \frac{25(3+4\varepsilon)}{(5+8\varepsilon)^2}$.

It is observed that for agent 1, the contract is Pareto-optimal. For the second agent the effort level decreases with increasing parameter ε.

Graph 2. *Optimal contract*



The analysis can be extended to salaries, profit and information income.

Conclusions

The study of the literature, primarily mathematical and economic, on which this article is based leads to some theoretical and practical conclusions. Thus, first of all, these contracts, which are concluded in the free market, must offer the participant at least optimal, if not high, efficiency.

The free market offers different options and those who intend to achieve optimal contracts must rely on a number of information. This is a second conclusion that an optimal contract can only be realized on the basis of market dominance and in this sense the relevant information that gives meaning to the contractual transaction that is performed.

The study considered two entities (economic agents) that had a different volume of information. Starting from the grounded theories of Kuhn-Tucker, von Neumann-Morgenstern and the Pareto model, we went step by step to substantiate a tested mathematical model, which would ensure the realization of contracts in optimal conditions.

Another conclusion is that in the conditions of Big Data, the era we are currently going through, there is a huge volume of information, which can be used but with discernment to substantiate the conditions under which an optimal contract is concluded. At the same time, the Blockchain theory acquires an increasingly important place in the context of Big Data, which responds to the study concerns in order to conclude an optimal contract.

Although the article is very mathematical, it aimed to substantiate and explain the model that the authors propose as one that helps to substantiate and rent optimal contracts.

A final conclusion is that this model can be extended, it can be resistematized, so as to meet all the requirements regarding the substantiation and negotiation of optimal contracts.

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